Wandering Domains of Entire Functions

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Dedicated to my Parents and Teachers...

Abstract

In the first part of the report, we study the iteration theory for entire functions. We discuss multiply connected domains in the Fatou set of entire function. Then we construct a function in terms of canonical product, which have at least one multiply connected Fatou component. Afterwards we apply this example for polynomials and see its dynamics. Finally we find out differences between polynomials and entire function.

In the second part of this report, we study some properties of maximum modulus function for the transcendental entire functions. Then we study the dynamical behavior of a transcendental entire functions in multiply connected wandering domain. We introduce certain positive harmonic function h in a multiply connected wandering domain U , related to the harmonic measure. It provides us large details about a multiply connected wandering domains under the function f. Using this technique, we see that, for sufficiently large n, there exist annuli C_n contained in the forward image domains U_n and the union of these annuli works as an absorbing set for the iterates of f. In fact f behaves like a monomial within each such annuli. Lastly we discuss little bit on the proximity of ∂U_n and ∂C_n for large n.

Key words - Julia and Fatou Set, connected component, completely invariant, transcendental entire function, harmonic function, potential theory, omitted value, singular value, maximum modulus function.

Contents

1 Introductions and Definitions

Dynamics is the study of changes in a system with time. The complex dynamics deals with the iteration of a complex valued function. It may be a rational or a transcendental function. Rational functions are analytic on extended complex plane $\hat{\mathbb{C}}$ and the transcendental functions is defined in the whole complex sphere with an essential singularity at infinity. We say that dynamics is defined by the iteration of function over a set or a system. The iteration theory of of rational or entire function $f(z)$ of the complex variable z treats the sequence of "*iterates*"$\{f^n(z)\}$ defined by $f^0(z) = z$ $f^1(z) = z_1$ ..... $f^{n+1}(z) = f^1(f^n(z)), \qquad n = 0, 1, 2, \ldots$

1.1 Normality

Let $\{f^n\}$ be the iterates of the function of f. The family $\{f^n\}$ of functions defined on the plane is said to be normal at $z \in \mathbb{C}$ if every sequence extracted from $\{f^n\}$ has a subsequence which converges uniformly either to a bounded function or to ∞ on each compact subset of some neighborhood of z[12]. The iteration of function f at point z is said to behave chaotically if we make an arbitrarily small perturbation in the z, it can cause large changes in the sequence of iterated function values. i.e., the iteration of f depends on the initial point.

1.2 The Fatou and the Julia set

Fatou set $F(f)$ of function f is the collection of all point $z \in \mathbb{C}$ for which the iterates behave normally in a neighborhood.

$$F(f) = \{z \in \mathbb{C} : \ \{f^n(z)\} \, is \ normal \ at \ z\}$$

The Julia set $J(f)$ of function f is the set of all $z \in \mathbb{C}$ where the iteration of function behaves chaotically in a neighborhood. Julia set also is defined the complement of Fatou set.

$$J(f) = \mathbb{C} - F(f)$$

1.3 Properties of the Julia and Fatou set

A subset A of the complex plane is completely invariant with respect to a function $f(z)$ if $w \in A$ implies $f(w) \in A$ and $w* \in A$ whenever $f(w*) = w$. In other words $f(A) = A = f^{-1}(A)$.

Proposition 1. *The Julia set $J(f)$ and Fatou set $F(f)$ are completely invariant. More precisely, $f^{-1}(J(f)) = f(J(f)) = J(f)$ and $f^{-1}(F(f)) = f(F(f)) = F(f)$.*

The proof can be found in [11].

Definition : (**Periodic point**) A point $z_0 \in \mathbb{C}$ is called a periodic point of a function f if there is a $n \in \mathbb{N}$ such that $f^n(z_0) = z_0$ and $f^p(z_0) \neq z_0$ for $0 \leq p < n$. If $n = 1$, then z_0 is called fixed point.

The periodic points are of following types :

1. if $|(f^n)'(z_0)| < 1$, it is said to be attracting ,(if$|f'(z_0)| = 0$ then it is called super attracting)
2. if $|(f_n)'(z_0)| = 1$ is said to be indifferent.
    - rationally indifferent if θ is rational in the expression $(f^n)'(z_0) = e^{2\pi i\theta}$
    - irrationally indifferent if θ is irrational in the expression $(f^n)'(z_0) = e^{2\pi i\theta}$
3. if $|(f_n)'(z_0)| > 1$, it is said to be repelling periodic

Definition : (**Set of Escaping points For** f)

The set $I(f) = \{z \in \mathbb{C} : \lim_{n\to\infty} f_n = \infty\}$ is called the set of escaping points of function f and the complement of $I(f)$ is known as filled Julia set of f. It is denoted by $K(f)$ and given as $K(f) = \mathbb{C} - I(f)$. The boundary of $K(f)$ is called Julia set, i.e; $J(f) = \partial K(f)$ If the function f is a polynomial, the set of escaping points $I(f)$ contains the point at infinity. Then Julia set is the boundary of escaping set.

1.4 Some other description of the Julia set

Here we enumerate few descriptions of the Julia set. $J(f)$ is the smallest closed set containing at least three points which is completely invariant under $J(f)$ is the closure of the set of repelling periodic points. If f is an entire function, then $J(f)$ is the boundary of the set of points which converge to infinity under iteration.i.e. the Julia set is the boundary of basin of infinity. If f is a polynomial, then $J(f)$ is the boundary of the filled Julia set; that is, those points whose orbits under iterations of f remain bounded.

1.5 Fatou Component

A subset D of the Fatou set $F(f)$ is called a Fatou component if the subset D is open connected in the Fatou set and there does not exist an open set D' such that $D \subset D' \subset F(f)$, i.e; maximal open connected subset of $F(f)$ is called the Fatou component.

For example: $f(z) = z^2$, the Fatou set of $f(z)$, is $\mathbb{C} - \{z \in \mathbb{C} : |z| = 1\}$, has two components (i) Open unit disk and, (ii) complement of closed unit disk.

1.6 Classification of Fatou Components

Let D be a Fatou component of f. As the Fatou $F(f)$ is completely invariant, therefore $f^n(D)$ is contained in a Fatou component of $F(f)$ which is denoted by D_n. A component D_n is called periodic if there is $n \in \mathbb{N}$ such that $f^n(D) \subset D$ and it is called preperiodic if there exist $n > m \geq 0$ such that $f^n(D) = f^m(D)$. If D is a periodic component with period n, then

$$\{D, D_1, D_2, ..., D_{n-1}\}$$

called a periodic cycle of component. If $n = 1$, then D is called invariant component of F(f). (classification theorem of periodic components[11]) Let D be a periodic component of f with period n, one and only of the following conditions is satisfied

1. *Attracting Basin*: If there is a point $z_0 \in D$ with $f_n(z_0) = z_0$ and $|(f_n)'(z_0)| < 1$ and every point $z \in D$ satisfies $f_{n_k}(z) \to z_0$ as $k \to \infty$. The point z is called attracting periodic point and the domain or component D is attracting basin.

2. *Parabolic basin*: There exist a point $z_0 \in \partial D$ with $f_n(z_0) = z_0$ and $(f_n)'(z_0) = e^{2\pi i\theta}$ where $\theta \in \mathbb{Q}$ and every point $z \in D$ satisfies $f_{n_k}(z) \to z_0$ as $k \to \infty$. The point z is called parabolic periodic point and the domain or component D is parabolic basin.

3. *Siegel Disc*: There exist a point $z_0 \in \partial D$ with $f_n(z_0) = z_0$ and $(f_n)'(z_0) = e^{2\pi i\theta}$ where $\theta \in \mathbb{R}\backslash\mathbb{Q}$, $f_n|D$ is conjugate an irrational rotation of an unit disk. The domain or component D is Siegel disc.

4. *Herman ring*: If there exists an analytic homeomorphism $\phi : D \to A$, A is the annulus$\{z : 1 < |z| < r\}, r > 1$, such that $\phi(f_k(\phi^{-1}(z))) = e^{i2\pi\theta}z$ for some $\theta \in \mathbb{R}\backslash\mathbb{Q}$, then D is called as Herman ring.

5. *Baker domains*: For $z \in D$ $f_n(z) \to \infty$ *as* $k \to \infty$ the domain D is called Baker domain.

Proposition 2. *Let f be a meromorphic function, and let D be an invariant component of F. Then the connectivity of D has one of the values 1, 2, or ∞. Here 2 occurs only when D is a Herman ring [11].*

2 Existence of Multiply Connected Fatou Component

In the theory developed by Fatou[4, 1] and Julia[12], a fundamental role is played by the Julia set $\mathbf{J}$(f) of those points of the complex plane where the family $\{f_n\}$ is not normal and the complement of it is known as Fatou set. In generally the Julia set has a very complicated shape so it is very difficult to determine the Julia set so as Fatou component. As Topfer has investigate that the entire function sin(z) and cos(z)

have all simply domains. Therefore I.N. Baker, keeping these result in his mind, he construct a function which has some multiply connected domains.

Proposition 3. : *The function $f(z)$ of (1) has at least one multiply connected domain D i.e the Fatou set of f(z) has at least one multiply connected component.*

In this report $f(z)$ will considered as non linear entire function. D_i are the components of Fatou set $F(f)$.

Proposition 4. : *If D_1 is multiply connected then $\lim_{n\to\infty} f_n(z) = \infty$ in D_1.*

Proposition 5. : *If D_1 is unbounded, then all D_i other than D_1 are simply connected. Moreover If D_1 is multiply connected then it is completely invariant.*

Lemma 2.1. : *There is an entire function $f(z)$ given by canonical product*

$$f(z) = cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right) \qquad 1 < r_1 < r_2 < ...., c > 0 \tag{1}$$

which satisfies

$$|f(e^{i\theta})| < \frac{1}{4} \qquad 0 \le \theta \le 2\pi \tag{2}$$

$$r_{n+1} < f(r_n) < 2r_{n+1} \qquad for\ all \qquad n = 1, 2, 3, ... \tag{3}$$

Proof. : Choose r_1 and $c > 0$ so that

$$c \exp\left(\frac{2}{r_1}\right) < \frac{1}{4} \qquad ; cr_1 > 1r_1 > 1. \tag{4}$$

Let us define the sequence $\{r_n\}_{n=1}^{\infty}$ inductively by $r_2 = cr_1 \left(1 + \frac{r_1}{r_1}\right) = 2cr_1^2$ and in general

$$r_{n+1} = cr_n^2 \left(1 + \frac{r_n}{r_1}\right)\left(1 + \frac{r_n}{r_2}\right)\left(1 + \frac{r_n}{r_3}\right)\dots\left(1 + \frac{r_n}{r_n}\right) \qquad n = 1, 2, 3, ... \tag{5}$$

Then $r_2 = cr_1^2 \left(1 + \frac{r_1}{r_1}\right) = 2cr_1^2$. Since $r_1 > 1$ so $r_2 > 2r_1$ and inductively $r_{n+1} > 2r_n$. From Equation (5) it is clear that each term $\left(1 + \frac{r_n}{r_i}\right)$ is greater than 1 so that $r_{n+1} \ge cr_n^2 > 2cr_1r_n > 2r_n$. Here it is clear that the distance between consecutive

terms increases with large quantity as n increases, thus $1 < r_1 < r_2 < ....$ holds. We know that

$$r_{n+1} > 2r_n \qquad \Longrightarrow \qquad r_{n+1} > 2^n r_1 \qquad n = 1, 2, 3, ... \qquad r_{n+k} = 2^k r_n. \tag{6}$$

By Weierstrass Factorization theorem[3], the function $f(z) = cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right)$ is entire if it is uniformly convergent.

Let $z = Re^{i\theta}$ for $0 \geq \theta \geq 2\pi$ and R very large real number. Then

$$|f(Re^{i\theta})| = cR^2 |\prod_{n=1}^{\infty} \left(1 + \frac{Re^{i\theta}}{r_n}\right)|$$

$$\leq cR^2 \prod_{n=1}^{\infty} \left(1 + |\frac{Re^{i\theta}}{r_n}|\right) = cR^2 \prod_{n=1}^{\infty} \left(1 + \frac{R}{r_n}\right). \tag{7}$$

In Equation (7), assume $T = cR^2 \prod_{n=1}^{\infty} \left(1 + \frac{R}{r_n}\right)$,

$$\ln T = \sum_{n=1}^{\infty} \ln \left(1 + \frac{R}{r_n}\right)$$

is convergent if $\sum_{n=1}^{\infty} \frac{R}{r_n}$ is convergent. Therefore

$$|\sum_{n=1}^{\infty} \frac{R}{r_n}| \leq \frac{R}{r_1} \sum_{1}^{\infty} \frac{1}{2^{n-1}} = \frac{2}{r_1} \qquad \textit{since } r_n < 2^{n-1} r_1.$$

Thus this gives $T \leq e^{\frac{2R}{r_1}}$. From Equation (7)

$$|f(z)| \leq cR^2 \prod_{n=1}^{\infty} \left(1 + \frac{R}{r_n}\right) < cR^2 e^{\frac{2R}{r_1}}. \tag{8}$$

This shows that given function $f(z)$ is uniformly convergent in a disc with radius R and center at origin. Since R is arbitrary large, therefore given function is uniformly convergence in whole complex plane. Hence the function $f(z)$, as given in (1) is entire. Now taking $R = 1$ in Equation (8) this gives

$$|f(e^{i\theta})| \leq c \prod_{n=1}^{\infty} \left(1 + \frac{1}{r_n}\right) < ce^{\frac{2}{r_1}}$$

first inequality of Equation (4),

$$|f(e^{i\theta})| \leq c \prod_{n=1}^{\infty} \left(1 + \frac{1}{r_n}\right) < ce^{\frac{2}{r_1}} < \frac{1}{4}$$

$$|f(e^{i\theta})| < \frac{1}{4}.$$

Further

$$r_{n+1} = cr_n^2 \prod_{k=1}^{n}\left(1+\frac{r_n}{r_k}\right) < f(r_n) = cr_n^2 \prod_{k=1}^{\infty}\left(1+\frac{r_n}{r_k}\right) = r_{n+1} \prod_{k=n+1}\left(1+\frac{r_n}{r_k}\right)$$

$$r_{n+1} < f(r_n) = r_{n+1} \prod_{k=n+1}^{\infty}\left(1+\frac{r_n}{r_k}\right).$$

But from second part of Equation (6)

$$\prod_{k=n+1}^{\infty}\left(1+\frac{r_n}{r_k}\right) < \prod_{k=1}^{\infty}\left(1+\frac{1}{2^k}\right) \leq 1 + \sum_{k\geq 2}\frac{1}{2^k} = 2.$$

Therefore

$$r_{n+1} < f(r_n) = r_{n+1} \prod_{k=n+1}^{\infty}\left(1+\frac{r_n}{r_k}\right) < 2r_{n+1}$$

$$r_{n+1} < f(r_n) < 2r_{n+1}.$$

Hence equation (2) and (3) is proved. □

Observations : From the first part of this lemma it is clear that the unit circle is mapped into a disc having radius $\frac{1}{4}$ and center at origin by the function $f(z)$. The second result of lemma shows that the function $f(z)$ pushes a circle with radius r_n and center ot the origin inside the annulus $A(r_{n+1}, 2r_n+1)$ for all $n = 1, 2, 3, ...$ Since each r_i is greater than 1, So repeated iteration of f, r_n converges to infinity.

Lemma 2.2. *: If $f(z)$ is a function as given in Lemma 1, then*

$$f(r_n^{\frac{1}{2}}) < r_{n+1}^{\frac{1}{2}} \qquad n = 1, 2, 3, ... \textit{ and} \tag{9}$$

$$\frac{1}{4}f(r_n^2) > r_{n+1}^2 \qquad n = 1, 2, 3, ... \tag{10}$$

Proof. : We have $f(z) = cz^2\prod_{n=1}^{\infty}\left(1+\frac{z}{r_n}\right)$, letting $|z| = r$ $|f(z)| = |cz^2\prod_{n=1}^{\infty}\left(1+\frac{z}{r_n}\right)| < cr^2\prod_{n=1}^{\infty}\left(1+\frac{r}{r_n}\right) < f(r)$. $f(r)$ is maximum modulus of $f(z)$ for $|z| = r$. Applying Hadamard's convexity Theorem to $V(s) = \log(f(e^s))$, we obtain for $s > 0$

$$V(2s) - V(0) > 2V(s) - V(0)$$

$$V(2s) > 2V(s) - V(0)$$

$$\log(f(e^{2s})) > 2\log(f(e^s)) - \log(f(e^0)) > 2\log(f(e^s)) - \log(f(1)$$

$$\log(f(e^{2s})) > \log\left(\frac{(f(e^s))^2}{f(1)}\right)$$

Taking antilogarithm, we have $f(e^{2s}) > \frac{(f(e^s))^2}{f(1)}$; since $f(1) < \frac{1}{4}$ so that

$$f(e^{2s}) > \frac{(f(e^s))^2}{f(1)} > 4(f(e^s))^2. \tag{11}$$

Now let $z = e^s \quad i.e;\ |z| = r$

Equation (11) gives

$$f(r^2) > \frac{(f(r))^2}{f(1)} > 4f(r)^2. \tag{12}$$

Put $r = r_n^{\frac{1}{2}}$ in Equation (12) and using (3) gives

$4f(r_n^{\frac{1}{2}})^2 < f(r_n) < 2r_{n+1}$ i.e;

$2f(r_n^{\frac{1}{2}}) < r_{n+1}^{\frac{1}{2}}$ i.e; $f(r_n^{\frac{1}{2}}) < r_{n+1}^{\frac{1}{2}}$ which proves (9). Putting $r = r_n$ in Equation (12) and using Equation (3) gives

$$f(r_n^2) > 4f(r_n)^2 > 4r_{n+1}^2$$

$\frac{1}{4}f(r_n^2) > r_{n+1}^2$ which proves Equation (10). Hence proof of Lemma 2 is complete. □

Lemma 2.3. *If $f(z)$ is the function as given in (1), then*

$$f(r) < 4|f(-r)| \tag{13}$$

holds in the region

$$B_n := \left\{r : 4r_n < r < \frac{1}{4}r_{n+1}\right\} \tag{14}$$

for all large enough n.

Proof. : Since we have

$r_{n+1} = cr_n^2 \left(1+\frac{r_n}{r_1}\right)\left(1+\frac{r_n}{r_2}\right)\left(1+\frac{r_n}{r_3}\right)\dots\left(1+\frac{r_n}{r_n}\right) \qquad n = 1,2,3,....$

$\frac{r_{n+1}}{r_n} > cr_n \to \infty \qquad as \qquad n\to\infty,$ since $r_n \to \infty$ as $n \to \infty$ so that B_n is non-empty for all sufficiently large n. We note that

$$\log(1+x) < x \qquad for \qquad x > 0$$

$$-\log(1-x) < 2x \qquad for \qquad 0 < x < \frac{1}{2}.$$

On adding these to inequalities we get

$$\log\left(\frac{1+x}{1-x}\right) < 3x \qquad for \qquad 0 < x < \frac{1}{2} \tag{15}$$

since $f(r) = cr^2 \prod_{n=1}^{\infty}\left(1+\frac{r}{r_n}\right)$, then

$$\log|f(r)| = \log(cr^2) + \sum_{n=1}^{\infty} \log|1+\frac{r}{r_n}|$$

and

$$\log|f(-r)| = \log(cr^2) + \sum_{n=1}^{\infty} \log|1-\frac{r}{r_n}|.$$

These gives

$$\log\left|\frac{f(r)}{f(-r)}\right| = \sum_{n=1}^{\infty} \log\left|\frac{1+\frac{r}{r_n}}{1-\frac{r}{r_n}}\right|$$

$$\log\left|\frac{f(r)}{f(-r)}\right| = \sum_{n=1}^{\infty} I_n = \sum_{k=1}^{n-1} I_k + I_n + I_{n+1} + \sum_{k=n+2}^{\infty} I_k \tag{16}$$

where

$$I_n = \log\left|\frac{1+\frac{r}{r_n}}{1-\frac{r}{r_n}}\right|. \tag{17}$$

For r belongs to the annulus B_n for $k \le n-1$, then $r_k < r_{n-1}$ and $4r_n < r < \frac{1}{4}r_{n+1}$ these gives $0 < \frac{r_k}{r}$. Since $0 < r_k < r$ and $r > 4r_n$ i.e; $\frac{1}{r} < \frac{1}{4r_n}$,

$$0 < \frac{r_k}{r} < \frac{r_k}{4r_n} < \frac{r_{n-1}}{4r_n} < \frac{r_{n-1}}{4(2r_{n-1})} < \frac{1}{8}.$$

By the equation (15) and (17) we get

$$0 < I_k = \log\left|\frac{1+\frac{r}{r_n}}{1-\frac{r}{r_n}}\right| < 3\frac{r_k}{r} < \frac{3r_k}{4r_n}.$$

Hence

$$\sum_{k=1}^{n-1} I_k < \frac{3}{4}\sum_{k=1}^{n-1}\frac{r_k}{r_n} = \frac{3}{4r_n}\sum_{k=1}^{n-1} r_k$$

$$= \frac{3}{4r_n}\{r_1 + r_2 + r_3 + ... + r_{n-1}\}$$

$$= \frac{3r_{n-1}}{4r_n}\left\{1 + \frac{r_{n-2}}{r_{n-1}} + \frac{r_{n-3}}{r_{n-1}} + ... + \frac{r_1}{r_{n-1}}\right\}$$

$$= \frac{3r_{n-1}}{4r_n}\left\{1 + \frac{1}{2^1} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}\right\}$$

$$= \frac{3r_{n-1}}{4r_n}\left\{1 + \frac{1}{2^1} + \frac{1}{2^2} + ...\right\} < \frac{3}{2}\frac{r_{n-1}}{r_n}. \tag{18}$$

Now for $k > n+2$, $r_k \geq r_{n+2}$ *i.e*; $\frac{1}{r_k} \leq \frac{1}{r_{n+2}}$ and $r \in B_n$ i.e; $r < \frac{1}{4}r_{n+1}$ so that

$$0 < \frac{r}{r_k} < \frac{r_{n+1}}{r_k} < \frac{r_{n+1}}{4r_{n+2}} < \frac{1}{8}.$$

By Equations (15) and (17), we get

$$0 < I_k = \log\left|\frac{1+\frac{r}{r_n}}{1-\frac{r}{r_n}}\right| < 3\frac{r}{r_k} < \frac{3r_{n+1}}{4r_k}.$$

Hence

$$\sum_{k=n+2}^{\infty} I_k < \frac{3}{4}\sum_{k=n+2}^{\infty}\frac{r_{n+1}}{r_k} = \frac{3r_{n+1}}{4}\sum_{k=n+2}^{\infty}\frac{1}{r_k}$$

$$= \frac{3r_{n+1}}{4}\left\{\frac{1}{r_{n+2}} + \frac{1}{r_{n+3}} + \frac{1}{r_{n+4}} + ...\right\}$$

$$= \frac{3r_{n+1}}{4r_{n+2}}\left\{1 + \frac{r_{n+2}}{r_{n+3}} + \frac{r_{n+2}}{r_{n+4}} + ...\right\}$$

$$= \frac{3r_{n+1}}{4r_{n+2}}\left\{1 + \frac{1}{2^1} + \frac{1}{2^2} + ...\right\} < \frac{3r_{n+1}}{2r_{n+2}}. \tag{19}$$

Applying equation (18) and (19) in (16) and it follows that for r satisfying (14)

$$\log\left|\frac{f(r)}{f(-r)}\right| < \frac{3}{2}\frac{r_{n-1}}{r_n} + \frac{3}{2}\frac{r_{n+1}}{r_{n+2}} + \log\left|\frac{1+\frac{r}{r_n}}{1-\frac{r}{r_n}}\right| + \log\left|\frac{1+\frac{r}{r_{n+1}}}{1-\frac{r}{r_{n+1}}}\right|$$

$$< \frac{3}{2}\frac{r_{n-1}}{r_n} + \frac{3}{2}\frac{r_{n+1}}{r_{n+2}} + \log\left(\frac{5}{3}\right) + \log\left(\frac{5}{3}\right), \quad since\frac{r}{r_n} < \frac{1}{4}\ and\ \frac{r}{r_{n+1}} < \frac{1}{4}. \tag{20}$$

As we have above mentioned that $\frac{r_{n+1}}{r_n} \to \infty$ *as* $n \to \infty$ so that for sufficiently large n Equation (20) gives

$$\log \left| \frac{f(r)}{f(-r)} \right| < \log 4$$
$$\frac{f(r)}{f(-r)} < 4$$
$$f(r) < 4f(-r).$$

Hence lemma is proved. □

Theorem 2.1. *: If $f(z)$ is the function as given in* (1) *and A_n is the annulus*

$$A_n = \left\{ z : r_n^2 < |z| = r < r_{n+1}^{\frac{1}{2}} \right\} \tag{21}$$

then there is an integer $N > 0$ such that $\forall n > N$ the mapping $z \mapsto f(z)$ maps A_n into A_{n+1} and $f_n(z) \to \infty$ uniformly in A_n. For each $n > N$, A_n belongs to a multiply connected component of $F(f)$.

Proof. : We have already seen that
$r_{n+1} = cr_n^2 \left(1 + \frac{r_n}{r_1}\right) \left(1 + \frac{r_n}{r_2}\right) \left(1 + \frac{r_n}{r_3}\right) \ldots\ldots.. \left(1 + \frac{r_n}{r_n}\right).$
Then any fixed m, $\frac{r_{n+1}}{r_n^m} \mapsto \infty$ thus the annuli A_n are non-empty for sufficiently large n. If $r_n \geq 4$ and $r_{n+1} \geq 16$

$$B_n : 4r_n < r < \frac{1}{4} r_{n+1} \;\; and A_n : 4r_n^2 < |z| = r < r_{n+1}^{\frac{1}{2}},$$

$$since \quad r_n > 4 \quad so \quad r_n^2 > 4r_n \;\; and \;\; r_{n+1} > 16 \;\; i.e; \;\; r_{n+1}^2 > 16 r_{n+1}.$$

Therefore $\qquad 4r_n < r_n^2 < |z| = r < r_{n+1}^{\frac{1}{2}} > \frac{1}{4} r_{n+1}.$

Hence A_n lies inside the annulus B_n. let $N > 0$, for all $n > N$ and $z \in A_n$, we have from Lemma 2, $f(r_n^{\frac{1}{2}}) < r_{n+1}^{\frac{1}{2}}$ and $\frac{1}{4} f(r_n^2) > r_{n+1}^2$ this gives that

$$|f(z)| \leq f(|z|) < f(r_{n+1}^{\frac{1}{2}}) < r_{n+1}^{\frac{1}{2}} \tag{22}$$

Since we know that $A_n \subset B_n$. So by Lemma 3 i.e; $f(r) < 4f(-r)$

$$|f(z)| \geq f(-|z|) > \frac{1}{4} f(|z|) > \frac{1}{4} f(r_n^2) > r_{n+1}^2. \tag{23}$$

On combining Equations (22) and (23) we get $r^2_{n+1} < f(|z|) < r^{\frac{1}{2}}_{n+1}$ this shows that A_n is mapped into A_{n+1} by $f(z)$. Now applying iterates of f on A_n as

$f(A_n) \subset A_{n+1}$

$f(f(A_n)) \subset f(A_{n+1}) \qquad$ i.e.; $f_2(A_n) \subset A_{n+2}$

...

$f_p(A_n) \subset A_{n+m}$.

Thus A_n is mapped into A_{n+m} by $f_m(z)$ and the minimum distance of A_n from $z = 0$ is r^2_{n+m}

$$\lim_{m\to\infty} r^2_{n+m} = \infty,$$

we have $\lim_{m\to\infty} f_m = \infty$ uniformly in A_n. Let F_n be the Fatou component which contains A_n. Now we see, in unit disk $|z| \le 1$ has $f(z) < \frac{1}{4}$. By Schwartz's lemma we get $f(z) < \frac{1}{4}|z|$; thus iteration theory gives $f_n(z) < 4^{-n}|z|$ and

$$\lim_{n\to\infty} f_n(z) = 0$$

uniformly in the unit disk, which belongs to some Fatou component F_0 (F_0 is the super attracting basin of the origin). Hence every F_n is multiply connected because any two different attraction of basin can not intersect. Thus F_n's are pair-wise disjoint [[11], *Theorem* 3.1.5]. Hence F_n is the multiply connected domain. □

2.1 The construction applied to polynomials

We have $f(z) = cz^2 \prod_{n=0}^{\infty} \left(1 + \frac{z}{r_n}\right)$ and let k be an integer greater than N of the previous theorem. Let $P(z)$ be the finite canonical product given by

$$P(z) = cz^2 \prod_{n=0}^{k} \left(1 + \frac{z}{r_n}\right). \tag{24}$$

For all z with $|z| < r_{k+1}$ we have

$$|f(|-z|)| \le P(-|z|) \le |P(z)| \le P(|z|) \le f(|z|).$$

This gives for $n = N+1, N+2, ..., k$ the annulus A_n of previous theorem into annulus A_{n+1} by $P(z)$. In particularly A_k is mapped into a region where $|z| > r^2_{k+1}$.

For $|z| = r_k^2$ belongs to A_k, then

$$|P(z)| \geq r_{k+1}^2$$

and

$$\left|\frac{P(z)}{z^2}\right| \geq \left(\frac{r_{k+1}}{r_k}\right)^2 \geq 4$$

Since we know that zeros of $P(z)$ are at $-r_1, -r_2, -r_3, ..., -r_k$ and modulus of zeros has at most $r_k < r_k^2$.

$|P(z)| \geq 4|z|^2 \geq 4|z|, \qquad \forall \ |z| > r_k^2$

Using iteration of P we have $|P_n(z)| \geq 4^n|z|^2 \geq 4|z|$. Thus

$$\lim_{n \to \infty} P_n(z) = \infty$$

uniformly in $|z| > r_{k+1}^2$ and also in annulus A_n for $n = N+1, N+2, ..., k$. Now $P(r) = cr^2 \prod_{n=0}^{k} \left(1 + \frac{r}{r_n}\right)$ i.e. $\frac{P(r)}{r} = rc \prod_{n=1}^{k} \left(1 + \frac{r}{r_n}\right)$. This shows that $\frac{P(r)}{r}$ is increasing function for $r \geq 0$, so that there is unique $R > 0$ such that $P(R) = R$ with for any $r > R$, it is $P(r) > r$. Thus if $r > R$ the sequence $P_n(r)$ is increasing and divergent. If it is convergent to s then $P(s) = s, s > R$, which is contradiction to the existence of unique R. After finite number of terms $P_n(r) \geq r_{k+1}^2$. Thus since $K = \left\{z : |z| > r_{k+1}^2\right\}$ in $F(P)$, and we know that $F(P)$ is be complete invariant so that the $r > R$ of real axis is in Fatou set of $F(P)$. In fact this ray and set K both belongs to same Fatou component. Now from equation (5) and (23) we obtained

$$P(r_1) \geq 2cr_1^2 > 2r_1$$

this gives $r_1 > R$. Therefore A_n, $n = N+1, N+2, ..., k$ must all belong to same component D_1 of $F(P)$ as the set K which are connected by the ray $r > R$. As $|P(z)| < |f(z)|$, we have $|P(e^{i\theta})| < 1/4$ and the unit circle belongs to a region of normality $D_0 \neq D_1$. Each of the zeros $-r_k$ is contained in a domain of normality D'_k where $\lim_{n \to \infty} P_n(z) = 0$ other than D_1. Thus D_1 is multiply connected in fact at least $(k - N)$ fold connected and the domains $D'_{N+1}, D'_{N+2}, D'_{N+3}, ..., D'_k, D_0$ are all different. In the above discussion, we follows the following result.

Proposition 6. : *The connectivity of D_1 is infinite.*

Proof. : Let us assume that the connectivity of D_1 is finite. So the boundary of D_1 consist of a finite number of disjoint components each compact and connected.
Let $d > 0$ be the minimum distance between different components. Thus Julia set $J(P)$ of P has finite number of component and since Julia set is perfect so each component contains an infinity of points. Let C_1 be any such boundary component. Let for any $s \in J(P)$, the disc $D : |z - s| < \rho\ ,\ \rho > 0$ and any bounded set E of the plane does not meets the neighborhoods N_1 *and* N_2 of two possibly exceptional points respectively. There is an n_0 such that $E \subset P_n(D)\ \forall n > n_0$. Let $E = C_1 - (N_1 \cup N_2) \neq \emptyset$ and taking radius ρ of D is so small that $\rho < d$. Then for $n > n_0$, $P_n(D)$ intersect C_1 therefore D contains the points of backward orbit P_{-n} and these all points belong to the boundary of Julia set $J(P)$and let these points belong to a component C_2(say)of $J(P)$ as s. Thus $P_n(C_2)$ meets C_1 for $n > n_0$ since $P_n(C_2)$ is connected subset of $J(F)$(as component)

$$P_n(C_2) \subset C_1 \qquad\qquad for\ \ n > n_0$$

Now s is arbitrary and D_1 is multiply connected. we assume that $C_1 \neq C_2$. we know that every $s \in J(P)$ is a limit point of fixed points in $J(P)$ i.e. points $z \in J(P)$ such that $P_m(z) = z$ for some integer m. Taking m to be sufficient large then $P_m(C_2) = C_2$. Hence that is the contradiction of $P(C_2) = C_1\ n > n_0$ and $C_1 \neq C_2$. Therefore the connectivity of D_1 is not finite. Thus D_1 is multiply connected with infinite fold. □

2.2 A difference between the transcendental and polynomial cases

It is interesting to note that, contrast to the case of $P(z)$ in (24) that the annuli A_n (21) are connected by a segment of the positive real axis belongs to $F(f)$.

Theorem 2.2. *There is a unique $R > 0$ such that $f(R) = R$, and for $r > R$ we have $f(r) > r$. There is $R' > 0$ such that $|f(re^{i\pi/2})| > 2r$ for $r > R'$. Then for any $r_1 > \max(R, R')$, the interval $[r_1, f(r_1)]$ contains a point of $J(f)$.*

Proof. As we know that the function $|f(z)|$ increases, so the function $\frac{1}{r}|f(re^{i\pi/2})|$ increases monotonically from 0 to ∞ as r increases from 0 to ∞. This provides the

existence of R' and so R. Define the function $\varphi(\theta)$ to be

$$\varphi(r,\theta) = \arg f(re^{i\theta}) = 2\theta + \sum_{n=1}^{\infty} \arg\left(1 + \frac{re^{i\theta}}{r_n}\right) > 2\theta$$

with all the arg function to zero at $\theta = 0$. For fixed r, $\varphi(r,\theta)$ is monotone increasing in $0 \leq \theta \leq \frac{\pi}{2}$, $\varphi(r,\theta)$ increases steadily to ∞ as r increases to ∞. For fixed $\alpha > 0$ put $\theta(r) = \theta(\alpha, r)$ is equal to the smallest positive solution θ of $\varphi(r,\theta) = \alpha$. Then $\theta(r)$ defined for all sufficiently large r and decreases steadily to 0 as $r \to \infty$.

We see that $|f(re^{i\theta(r)})| \to \infty$ monotonically as $r \to \infty$, since $r_1 < r_2,\ \theta < \pi/2$, then

$$|f(r_1 e^{i\theta(r_1)})| < |f(r_2 e^{i\theta(r_1)})| < |f(r_2 e^{i\theta(r_2)})|.$$

Here we give two applications of the function $\theta(\alpha, r)$:

(i) $\alpha = \pi$, as $r \to \infty$, $f(re^{i\theta(r)})$ goes to ∞ along the negative real axis and that runs through the value $-r_n$ for all sufficiently large n. As $\theta(r) \to 0$ as $r \to \infty$, we conclude that for any $\epsilon > 0$ the angle:

$$\{0 \leq \arg z < \epsilon,\ 0 \leq r < \infty\}$$

which we denote by A_ϵ, contains the curve $z = re^{i\theta(r)}$ for all sufficiently large r, hence A_ϵ contains w_n where $|w_n| \to \infty\ as\ n \to \infty$ such that $f(w_n) = -r_n,\ f_m(w_n) = 0$ for all $m \geq 2$.

(ii) Now we suppose that for some $r_1 > \max(R, R')$, the interval $[r_1, f(r_1)]$ belongs to $F(f)$. Since $F(f)$ is open, it must contain the set

$$W\{r_1, f(r_1), \epsilon\} = \{z : r_1 \leq |z| \leq f(r_1), 0 \leq \arg z < \epsilon\} \qquad where\ \epsilon > 0.$$

Take $\alpha = \epsilon < \pi/2$, in defining $\theta(r) = \theta(r, \epsilon)$. Now $\varphi(r,\theta) > 2\theta$ so that $\theta(\epsilon, r) < \epsilon/2$. Thus for $r_1 < r < f(r_1)$ the segment S_r defined as

$$0 \leq \arg z < \theta(r)$$

of the circle $|z| = r$ belongs to $W\{r, f(r_1), \epsilon\}$. Further since $|f(re^{i\theta})|$ is a decreasing function of θ, we see that $f(S_{r_1}) \subset W\{r_1, f(r_1), \epsilon\}$. For $r_1 \leq r \leq f(r_1)$ $f(S_r)$ is a

simple arc, whose minimum distance from the origin occurs at the upper end-point; for $r = f(r_1)$ the minimum occurs the

$$f\left\{f(r_1)\exp(i\theta(f(r_1)))\right\}$$

since $\theta < \pi/2$ so it has the value $> 2f(r_1)$. As r increases from r_1 *to* $f(r_1)$, the arc $f(S_r)$ sweep out a region which contains

$$W\left\{r_1, 2f(r_1), \epsilon\right\}$$

and we can see that S_r is contained in $F(f)$ and so that region which swept by the arc $f(S_r)$. Thus $F(f)$ contains $W\left\{r_1, 2f(r_1), \epsilon\right\}$. By the inductive repetition of above argument we obtain that $F(f)$ contains the whole angle A_ϵ. Therefore the angle A_ϵ combines with the A_n in the Theorem2.1 that gives

$$\lim_{n\to\infty} f_n(z) = \infty$$

but by (i) A_ϵ contains the point w_n at which $\lim_{n\to\infty} f_n(w_m) = 0$ This contradiction shows in the fact the interval $[r_1, f(r_1)]$ must contains points of $J(f)$. Hence proof is complete. □

3 Dynamics on Wandering Domains

Let U be a connected component of $F(f)$. Then $f^n(U) \subset U_n$, where U_n is a component of $F(f)$. If p is the smallest positive integer such that $U_p = U$, then U is called *periodic* of period p. In particular, if $p = 1$ then U is called invariant component. If for some integer $n \geq 1$, U_n is periodic, while U is not periodic, then U is called *preperiodic*. If all U_n are disjoint from each other, that is, for $m \neq n$, $U_m \neq U_n$ i.e., $f^n \cap f^m = \phi$ then U is called wandering domain. For a general transcendental entire function, periodic components can only be basin of attraction of attracting or parabolic cycles, Siegel disk or baker domains (also known as parabolic domains at infinity), depending on the limiting behavior $\{f^n\}$.

Definition : **(Singular Value)** A point z_0 is said to be critical point of the meromorphic function f if $f'(z_0) = 0$ and the value of $f(z_0)$ is called the critical value of f. A point a is called an asymptotic value of the function f if there exists a continuous curve $\gamma : [0, 1) \to \hat{\mathbb{C}}$ satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = a$. Here the curve γ is called an asymptotic path. The set of all critical and finite asymptotic values of function are known as *singular values* . The set of all singular values of f is denoted by S_f [14].

A function f is called critically bounded or function of bounded type if the set S_f is bounded. The class of all such functions is denoted by $\mathcal{B}$. In particular, the function f is called critically finite or function of finite type if the set S_f is finite. The class of all such functions is denoted by $\mathcal{S}$. It is clear that the class $\mathcal{S}$ contains in the class $\mathcal{B}$ i.e; $\mathcal{S} \subset \mathcal{B}$. The set of transcendental entire function are very large. Sullivan's, Eremenko, Lyubich, Goldberg and Keen, in their works, restricted the entire in to classes. These classes known as 'Speiser class $\mathcal{S}$ and Eremenko-lyubich class $\mathcal{B}$' [1, 10]

Definition : Special classes of Entire Functions

The class $\mathcal{S}$ ("Speiser class") consists of those entire functions that have only finitely many singular values. The class $\mathcal{B}$ ("the Eremenko-Lyubich class of functions of bounded type") consists of those entire functions for which all singular values are contained in a bounded set in $\mathbb{C}$.

For an entire function of class $\mathcal{B}$, all transcendental singularities over ∞ are logarithmic. An entire function f has finitely many critical points and finitely many asymptotic tracts if and only if f has the form $f(z) = c + \int P(z')\exp(Q(z'))dz'$ for polynomials P, Q. The Fatou components of functions in class $\mathcal{S}$ are simply connected and also we know that Fatou components of those entire functions, which contain in class $\mathcal{B}$, are simple connected. In spite of these function, there are many examples of entire functions with multiply connected domains. Now if U is any multiply connected domain of Fatou set of transcendental entire function f, then U is wandering domain, and has the following properties

(1) each U_n is bounded and multiply connected,

(2) there exists $N \in \mathbb{N}$ such that U_n and 0 lie in a bounded complementary component of U_{n+1}, for $n \geq N$

(3) $dist(U_n, 0) \to \infty \; as \; n \to \infty$

We know that $U_n = f^n(U), \; for \; n \in \mathbb{N}$, since U is bounded and thus $f^n : U \to U_n$ is proper (define later). In the next section, we define maximum modulus and minimum modulus function. we discuss some properties of transcendental entire functions.

3.1 The properties of the maximum/minimum modulus function

In this section we discuss about maximum modulus and minimum modulus function. The maximum modulus function is denoted by $M(r, f)$ and define as

$$M(r, f) := max\,\{|f(z)| : |z| = r\}$$

and minimum modulus function is denoted by $m(r, f)$ and define as

$$m(r, f) := \{|f(z)| : |z| = r\}$$

If f is a transcendental entire function, the following results occurs.

Lemma 3.1. *: Let f be transcendental entire function and suppose that $\epsilon > 0$. There exists $R = R(f) > 0$ such that if $r > R$ then there exists*

$$z^{'} \in \{z : r \leq |z| \leq r(1 + \epsilon)\}$$

with

$$|f^n(z')| \geq M^n(r,f), \qquad for\ n \in \mathbb{N},$$

and hence

$$M((1+\epsilon)r, f^n) \geq M^n(r,f), \qquad for\ \ n \in \mathbb{N}.$$

Theorem 3.1. *: Let f be transcendental entire function. Then there exists $R = R(f) > 0$ such that, for all $0 < c' < 1 < c$ and all $n \in \mathbb{N}$*

$$\log M(r^c, f^n) \geq c \log M(r, f^n), \qquad for\ r \geq R$$

and

$$\log M(r^{c'}, f^n) \leq c' \log M(r, f^n), \qquad for\ r \geq R^{1/c'}.$$

Before proving theorem, we can see that let f be transcendental entire function , Then the function $\phi(t) = \log M(e^t)$ satisfies the $\phi(t)/t^k \to \infty$ as $t \to \infty$ for all $k > 1$. $\phi(t)$ is convex function.

Proof. : Let us consider a function $\phi(t) = \log M(e^t), \quad t \in \mathbb{R}$. Since function f is transcendental, and from above properties of $\phi(t)$, we have $\phi(t)/t \to \infty$ as $t \to \infty$. Now we can take t_0 and t_1, with $t_1 \geq t_0 > 0$ so large that

$$\phi(t_0) > 0 \qquad and \qquad \frac{\phi(t)}{t} \geq \frac{\phi(t_0)}{t_0}, \qquad for\ t \geq t_1. \tag{25}$$

If $\phi'(t)$ is denoted the right derivatives of $\phi(t)$. Then by the convexity of ϕ and (25), we have

$$\phi'(t) \geq \frac{\phi(t) - \phi(t_0)}{t - t_0} \geq \frac{\phi(t)}{t} \qquad for\ t \geq t_1.$$

Let $\Phi(t) = \frac{\phi}{t}$, in order to prove we have to show that the derivative of function $\Phi(t)$ is positive. We have

$$\Phi'(t) = \left(\frac{\phi(t)}{t}\right)' = \frac{\phi'(t)}{t} - \frac{\phi(t)}{t^2} \geq 0 \qquad for\ t \geq t_1$$

This shows that Φ is increasing function, that is $\frac{\phi(t)}{t}$ is increasing for $t \geq t_1$. This gives directly for $c > 1$ *and* $t \geq t_1$ that $\phi(ct) \geq c\phi(t)$. In other way, for $t \geq t_1$ and any $c > 1$ we have

$$\log\left(\frac{\phi(ct)}{\phi(t)}\right) = \int_t^{ct} \frac{\phi'(s)}{\phi(s)} ds \geq \int_t^{ct} \frac{1}{s} ds$$

$\implies$

$$\log\left(\frac{\phi(ct)}{\phi(t)}\right) \geq \log c$$

$$\implies \frac{\phi(ct)}{\phi(t)} \geq c.$$

This inequality is the particular case of required result, for $n = 1$.

Now let us consider $\phi_n(t) = \log M(e^t, f^n)$, for $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Then $\phi_1 = \phi$ and, for all $n \in \mathbb{N}$ the function ϕ_n are convex. In order to the prove of the inequality in the first statement of the lemma we required to find T sufficiently large that

$$\phi_n(ct) \geq c\phi_n(t) \qquad for\ t \geq T,\ c > 1\ and\ n \in \mathbb{N}$$

We know that $\frac{\phi(t)}{t}$ is increasing function so there exists $t_0 > 0$ such that

$$\frac{\phi(t)}{t} > 2 \qquad and\ \frac{\phi(t)}{t}\ is\ increasing\ for\ t \geq t_0/2 \tag{26}$$

From Lemma3.1 and by the definition of $\phi(t)$ and $\phi_n(t)$, we obtain

$$M((1+\epsilon)r, f^n) \geq M^n((r, f).$$

Replace f by ϕ and choose ϵ such as $(1+\epsilon)r = e^t\ and\ r = e^{t/2}$ this gives that $M(e^t, \phi^n) \geq M^n(e^{t/2}, \phi)$. Now taking logarithm both side, we have

$$\phi_n(t) \geq \phi^n(t/2) \qquad t \geq t_0 \tag{27}$$

Let $\Phi(t) = \frac{\phi(t)}{t}$ is an increasing function as we have seen and also $\Phi \to \infty\ as\ t \to \infty$ so choose $T \geq 2T_0$ such that

$$\frac{\Phi(t)}{\Phi(t_0)} \geq 2 \quad \implies \quad \frac{\phi(t)}{t} \geq 2\frac{\phi(t_0)}{t_0} \qquad for\ t \geq T/2 \tag{28}$$

Then, for $t \geq T$, we have $\phi^n(t/2) \geq \phi(t/2) \geq t \geq T \geq t_0$ for $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{\phi_n(t)}{t} &\geq \frac{\phi^n(t/2)}{t} && using\ (27) \\
&\geq \frac{1}{2}\frac{\phi^n(t/2)}{t/2} \\
&= \frac{1}{2}\frac{\phi(\phi^{n-1}(t/2))}{\phi^{n-1}(t/2)}.\frac{\phi(\phi^{n-2}(t/2))}{\phi^{n-2}(t/2)}\cdots\frac{\phi(\phi(t/2))}{\phi(t/2)}.\frac{\phi(t/2)}{t/2}
\end{aligned}$$

$$= \frac{1}{2}\frac{\phi(\phi^{n-1}(t_0))}{\phi^{n-1}(t_0)}.\frac{\phi(\phi^{n-2}(t_0))}{\phi^{n-2}(t_0)}\cdots\frac{\phi(\phi(t_0))}{\phi(t_0)}2\frac{\phi(t_0)}{t_0}$$
$$\geq \frac{\phi^n(t_0)}{t_0} \geq \frac{\phi_n(t_0)}{t_0}.$$

Which give that $\phi_n(t)$ is monotonically increasing function. Hence we have $\phi_n(ct) \geq c\phi(t)$ for $c > 1$ this implies that

$$\log M(e^{ct}, f^n) \geq c \log M(e^t, f^n) \qquad for\ n \in \mathbb{N}.$$

Putting $e^t = r \geq R$ in above inequality we get

$$\log M(r^c, f^n) \geq c \log M(r, f^n) \qquad for\ n \in \mathbb{N}\ and\ r \geq R.$$

This proves first inequality. To prove second inequality we consider $0 < c' < 1 < c$, since $c' < 1$ so that $c't < t$ this implies $\phi_n(c't) \leq c'\phi(t)$ which gives the second inequality. Hence proof of Theorem is complete.

□

We get the useful case of Theorem 3.1, that is

$$\log M(2r, f^n) \geq (1 + \log 2/\log r) \log M(r, f^n) \tag{29}$$

3.2 Properties of functions which omitted values in an annulus

In this section, we study the entire function which defined on large annulus whose images omits certain values. We will see that if the image omits the unit disc, then the function behaves like a monomial inside the annulus and, under the weaker assumption that zero is omitted, we will find strong covering results by the using the hyperbolic metric.

Definition : A continuous map $f : U \to V$, where U and V are domains, is called *proper* if for each compact subset C of V the inverse image $f^{-1}(C)$ is also compact in U.

Let $f : U \to V$ be a proper analytic function. Then there exists a positive integer d, called the degree of f, such that each point in V has exactly d preimages in U,

counted according to multiplicity. In fact $f(U) = V$. The Riemann-Hurwitz formula states that for a proper analytic function f of degree d from U to V.

$$c(U) - 2 = d(c(V) - 2) + N$$

where N is the number of critical points of f in U and $c(U)\&c(V) < \infty$. Since we know fairly that f is an analytic function with degree d and number of critical points is N, counted according to multiplicity then N can not exceed to $2d - 2$. Thus N is finite always and also that $c(U) = \infty$ if and only if $c(V) = \infty$. Moreover let V_k be an increasing sequence of smoothly bounded domains compactly contained in V. Let U_k be an increasing sequence of components of the preimages of the V_k, and d_k is the degree of the proper map $f : U_k \to V_k$ and denoted by N_k the number of critical points of f in U_k. As every boundary component of U_k has only d_k preimages under f, we have $c(U_k) \leq d_k c(V_k)$. Now using the Riemann-Hurwitz formula to $f : U_k \to V_k$, we get

$$d_k c(V_k) \geq c(U_k) = d_k(c(V_k) - 2)N_k + 2 = d_k c(V_k) - 2d_k + N_k + 2$$

and thus $N_k \leq 2d_k - 2 \leq 2d - 2$. Let $c(U_k) = N'$, for sufficiently large N'. Since

$$c(U_k) \leq d_k c(V_k)$$

then $c(U_K) = N' \iff c(V_k) \geq \frac{c(U_k)}{d_k} = \frac{N'}{d_k}$
as N' is very large so let $N' \to \infty$, we have $c(U_k) = \infty \iff c(V_k) \geq \frac{c(U_k)}{d_k} = \frac{N'}{d_k} = \infty$. If an analytic function $f : U \to V$ extends continuously the the closer of U then f is proper if and only if $f(\partial U) = \partial V$. This implies that if U is bounded Fatou component of an entire function f, then $f^n : U \to f^n(U)$ is proper, for $n \in \mathbb{N}$. In fact this implies that U is multiply connected wandering domains of entire functions.

Theorem 3.2. *Let g be analytical in $A_n(r^a, r^b)$, for some $r > 1$, $0 < a < b$, and let*

$$m(\rho, g) > 1 \qquad for\ \rho \in (r^a, r^b). \tag{30}$$

(a) if $\epsilon \in (\pi/\log r, (b - a)/2)$, then

$$\log m(\rho, g) \geq \left(1 - \frac{2\pi}{\epsilon \log r}\right) \log M(\rho, g) \qquad for\ \rho \in [r^{a+\epsilon}, r^{b-\epsilon}].$$

(b) In particularly, if $\epsilon = 2\pi\delta$*, where* $\delta = 1/\sqrt{\log r} < min\{2, (b-a)/4\pi\}$*, then*

$$\log m(\rho, g) \geq (1-\delta)\log M(\rho, g) \qquad for\ \rho \in [r^{a+2\pi\delta}, r^{b-2\pi\delta}].$$

Proof. Let us consider $u(z) = \log|g(z)|$ is positive harmonic function in $A(r^a, r^b)$. We define a function $U(t) = u(e^t)$. Since $z \in A(r^a, r^b)$, then $r^a < |e^t| = |z| < r^b$ i.e. $a\log r < Re(t) < b\log r$.

Thus $U(t) = u(e^t)$ also is positive harmonic in strip

$$S = \{t : a\log r < Re(t) < b\log r\}$$

Now in oder to proof of part (a), since $\epsilon \in (\frac{\pi}{\log r}, \frac{(b-a)}{2})$ so that $\epsilon < \frac{b-a}{2} \implies 2\epsilon < b-a$ $\implies (\epsilon + a)\log r < (\epsilon - b)\log r$.

Choose t_1 & t_2 such as

$$(a+\epsilon)\log r \leq Re(t_1) = Re(t_2) \leq (b-\epsilon)\log r \qquad and \qquad |Im(t_1) - Im(t_2)| \leq \pi.$$

Then the $\{t : |t - t_1| < \epsilon\log r\} \subset S\ and\ |t_1 - t_2| \leq \pi < \epsilon\log r$. By the Harnack's inequality, i.e, if f is a positive harmonic function on Ω and K is a compact subset of Ω then there exist a constant $k = k(\Omega, K)$ (which depends on Ω *and* K) such that $f(z) \leq kf(z_0)$ *for all* $z, z_0 \in K$. Since $U(t)$ is positive harmonic function on $A(r^a, r^b)$ therefore there exists a constant $K(A, S)$ let us define as $K = \frac{\epsilon\log r + \pi}{\epsilon\log r - \pi}$, which is depend on annulus given in hypothesis and constructed strip, such that

$$U(t_2) \leq KU(t_1) \qquad for\ all\ t_1, t_2\ contain\ in\ strip$$

$$\implies \qquad \frac{U(t_2)}{U(t_1)} \leq \frac{\epsilon\log r + \pi}{\epsilon\log r - \pi} \qquad and \qquad \frac{U(t_1)}{U(t_2)} \leq \frac{\epsilon\log r + \pi}{\epsilon\log r - \pi}$$

$$\implies \qquad \frac{\epsilon\log r - \pi}{\epsilon\log r + \pi} \leq \frac{U(t_2)}{U(t_1)} \leq \frac{\epsilon\log r + \pi}{\epsilon\log r - \pi}.$$

Hence part (a) follows from the left-hand inequality and Part (b) follows from part (a). ∎

Theorem 3.3. *Let* f *be a transcendental function such that, for some* $n \in \mathbb{N}$, $r > 1$ *and* $0 < a < 1 < b$,

$$m(\rho, f^n) > 1, for \rho \in (r^a, r^b). \tag{31}$$

and suppose that $\delta = 1/\sqrt{\log r} < min\,\{1, (b-a)/(4\pi + 5\pi b)\}$.

There exists $R = R(f) > 1$ *such that if* $r^a \geq R$*, then the preimage under* f^n *of* $A(M(r^a, f^n), M(r^{(b-2\pi\delta)(1-\delta)}, f^n))$ *has a doubly connected component* A_n *such that*

$$A(r^{a+2\pi\delta}, r^{b(1-3\pi\delta)}) \subset A_n \subset A(r^a, r^b)$$

and f^n *has no critical point in* A_n*. Moreover, for* $r^a \geq R$*, we can write* $f^n(z) = P_n(\phi_n(z))$*,* $z \in A_n$ *where*

(a) ϕ_n *is conformal map on* A_n *which satisfies*

$$|z|^{1-\delta-2\pi\delta\log r/log|z|} \leq |\phi_n(z)| \leq |z|^{1+2\delta}, \qquad for\ r^{a+4\pi\delta} \leq |z| \leq r^{b(1-5\pi\delta)}; \tag{32}$$

(b) $P_n(z) = q_n z^{d_n}$*, where* $q_n > 0$ *and the degree* d_n *is the number of zeros of* f^n *in* $\{z : |z| < r\}$*, which satisfies*

$$(1-2\delta)\frac{\log M(r, f^n)}{\log r} \leq d_n \leq \frac{\log M(r, f^n)}{(1-a)\log r}.$$

Let us consider function $g = f^n$ and $\delta = 1/\sqrt{\log r}$ satisfy the hypothesis of Theorem 3.2 part (b). Also consider the annulus

$$C = A(r^{a+2\pi\delta}, r^{b(1-3\pi\delta)}). \tag{33}$$

Proposition 7. *Let* f *is transcendental entire function.If* $R = R(f) > 1$ *is constant suitable for Theorem 3.1, then for* $r^a \geq R$ *there is a double connected set* $A_n \supset C$ *such that* f^n *is a proper map of* A_n *onto the annulus*

$$A'_n = A(M(r^a, f^n), M(r^{(b-2\pi\delta)(1-\delta), f^n}))$$

Remark . *From the Riemann-Hurwitz formula we have that* f^n *has no critical points in* A_n*.*

Proof. To prove the above statement, first we will show that, as if $z \in C$

$$|f^n(z)| \leq M(r^{b(1-3\pi\delta)}, f^n) < M(r^{(b-2\pi\delta)(1-\delta)}, f^n) \qquad since\ b > 1$$

and

$$|f^n(z)| \geq M(r^{a+2\pi\delta}, f^n)^{1-\delta} \geq M(r^{(a+2\pi\delta)(1-\delta)}, f^n) > M(r^a, f^n) \qquad since\ 0 < a < 1.$$

By (31), Theorem 3.2 part (b) and Theorem 3.1, we have, for $r^a \geq R$, $f^n(C) \subset A'_n$. Now we define A_n to the component of $(f^n)^{-1}(A'_n)$ which contains C. Then

$$f^n(\{z : |z| = r^a\}) \subset \{z : |z| \leq M(r^a, f^n)\} \tag{34}$$

and

$$f^n(\{z : |z| = r^{b-2\pi\delta}\}) \subset \{z : |z| \geq M(r^{b-2\pi\delta}, f^n)^{(1-\delta)}\} \; by\; Theorem 3.2\; part\; (b)\; and\; Theorem 3.1$$

$$\subset \{z : |z| \geq M(r^{(b-2\pi\delta)(1-\delta)}, f^n)\} \qquad using\; (31). \tag{35}$$

Thus we have

$$C \subset A_n \subset A(r^a, r^{b-2\pi\delta}) \subset A(r^a, r^b). \tag{36}$$

Therefore from above discussion it is clear that A_n is doubly connected. This proves the Proposition. □

Proof of Theorem 3.3. Let us consider a conformal map ϕ_n of A_n onto an annulus of the form $A(r^a, r^{b'_n})$ where $b'_n > a$, with the property that the inner and outer boundary components of A_n map onto the inner and outer components of $A(r^a, r^{b'_n})$ respectively. $\phi_n(r)$ is positive. As $r^a \geq R > 1$, let $|z_1| = |z_2| = r^c$ then by Theorem3.2 part (b) and using (33), (36), we obtained,

$$|\phi_n(z_1)| = |\phi_n(z_2)|^{1-\delta} \qquad where\;\; a + 4\pi\delta \leq c \leq b(1 - 5\pi\delta)$$

we see that $b(1 - 5\pi\delta) > a + 4\pi\delta$ since $\delta < (b - a)/(4\pi + 5\pi b)$. Hence if $R_c = M(r^c, \phi_n),\; c > 0$, then

$$\phi_n(\{z : |z| = r^c\}) \subset \bar{A}(R_c^(1 - \delta), R_c) \qquad for\; a + 4\pi\delta \leq c \leq b(1 - 5\pi\delta). \tag{37}$$

We can see that the modulus of a ring domain is preserve under a conformal map and that is monotonic under containment, that the modulus of $A(r_1, r_2)$ is $(1/2\pi)(\log r_2/r_1)$[3]. We apply above statements to $A_n(c) = A_n \cap A(r^a, r^c)$, where $a+4\pi\delta \leq c \leq b(1-5\pi\delta)$, which is a ring domain by the (33) and (36). We can find, by (36) and (37), that

$$\frac{1}{2\pi} \log\left(\frac{r^c}{r^{a+2\pi\delta}}\right) \leq \quad \text{mod}\ (A_n(c)) \leq \frac{1}{2\pi} \log\left(\frac{r^c}{r^a}\right),$$

and

$$\frac{1}{2\pi}\log\left(\frac{R_c^{1-\delta}}{r^a}\right)\leq \quad \mathrm{mod}\ (\phi_n(A_n(c)))\leq \frac{1}{2\pi}\log\left(\frac{R_c}{r^a}\right).$$

Thus, since $0<\delta<1$,

$$r^{c-2\pi\delta}\leq R_c\leq r^{c(1+2\pi\delta)} \qquad for\ a+4\pi\delta\leq c\leq b(1-5\pi\delta) \tag{38}$$

Hence from (37) and (38), we have

$$r^{(c-2\pi\delta)(1-\delta)}\leq|\phi_n(z)|\leq r^{c(1+2\delta)}$$

which proves part (a).

To estimate the degree d_n we consider the mean

$$\mu(\rho,f^n)=\int_0^{2\pi}\log|f^n(\rho e^{i\theta})|d\theta \qquad \rho>0.$$

From (31) and Theorem3.2, we have

$$(1-\delta)\log M(\rho,f^n)\leq\mu(\rho,f^n)\leq\log M(\rho,f^n) \qquad for\ \rho\in[r^{a+2\pi\delta},r^{b-2\pi\delta}]. \tag{39}$$

Now we use the following subsequence of Jensen's Theorem

$$\mu(r,f^n)=\mu(\rho,f^n)+\int_\rho^r\frac{n(t)}{t}dt \qquad for\ \rho\in(0,r) \tag{40}$$

where $n(t)$ is the number of zeros of f^n in $\{z:|z|\leq t\}$. By the representation

$$f^n(z)=P_n(\phi_n(z))^{d_n}=q_n(\phi_n(z))^{d_n} \qquad for\ z\in A_n,$$

and the argument principle, we can get $n(t)\leq d_n$ for $0<t\leq r$ and $n(t)=d_n$ for $r^a\leq t\leq r$. Thus, by (40),

$$\mu(r^a,f^n)+(1-a)d_n\log r=\mu(r,f^n)\leq\mu(R,f^n)+d_n\log r; \tag{41}$$

We have $r^a\geq R=R(f)$, where $R>1$ is a suitable constant for Theorem3.1. Since, by (31), $\mu(r^a,f^n)\geq 0$, the equality in (41) and the right-hand inequality in (39) give

$$d_n\leq\frac{\mu(r,f^n)}{(1-a)\log r}\leq\frac{\log M(r,f^n)}{(1-a)\log r}. \tag{42}$$

If r is very large so that $\log R/\log r \leq 1/\sqrt{\log r} = \delta$, then by the right-hand inequality of (39) and Theorem 3.1, we have

$$\frac{\mu(R, f^n)}{\log r} \leq \frac{\log M(R, f^n)}{\log r} \leq \left(\frac{\log R}{\log r}\right) \frac{\log M(r, f^n)}{(1-a)\log r} \leq \delta \frac{\log M(r, f^n)}{\log r} \tag{43}$$

So, by the left-hand inequality in (39) and the inequality in (41) we get

$$(1-2\delta)\frac{\log M(r, f^n)}{\log r} \leq \frac{\mu(r, f^n) - \mu(R, f^n)}{\log r} \leq d_n.$$

Using (42) in the above inequality we get

$$(1-2\delta)\frac{\log M(r, f^n)}{\log r} \leq d_n \leq \frac{\log M(r, f^n)}{(1-a)\log r}.$$

Hence proof is complete. □

Now we are going to show that if an entire function omits the value 0 in an annulus, in fact more appropriately we say that there is omitting the unit disc, then the image of that annulus must cover a huge annulus which is much larger than that annulus as define in above theorem. In spite of this, we can not claim that the entire function behaves like a monomial with in the annulus as we discussed above. Now for proving further result we define density of hyperbolic metric.

Theorem 3.4. *There exists a absolute constant $\delta > 0$ such that if $f : A(R, R') \to \mathbb{C}\setminus\{0\}$ is analytic, where $R' > R$, the for all $z_1, z_2 \in A(R, R')$ such that*

$$\rho_{A(R,R')}(z_1, z_2) < \delta \qquad \textit{and} \qquad |f(z_2)| \geq |2f(z_1)| \tag{44}$$

we have

$$f(A(R, R')) \supset \bar{A}(|f(z_1)|, |f(z_2)|) \tag{45}$$

Proof. We will prove this theorem by making a contradiction. Let $A_0 = A(R, R')$ and also assume that the equation (45) holds for chosen $\delta > 0$. Suppose, for contradiction that $z_0 \in \bar{A}(|f(z_1)|, |f(z_2|)\setminus f(A_0)$. By the Schwartz-Pick lemma, any analytic self map f of Δ satisfies $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$, using this statement we have

$$\rho_{A_0}(z_1, z_2) \geq \rho_{f(A_0)}(f(z_1), f(z_2)) \geq \rho_{\mathbb{C}\setminus\{0,z_0\}}(f(z_1), f(z_2))$$

$$\implies \rho_{A_0}(z_1,z_2) \geq \rho_{\mathbb{C}\setminus\{0,1\}}(f(z_1)/z_0, f(z_2)/z_0).$$

Let us consider a hyperbolic geodesic γ in $\mathbb{C}\setminus\{0,1\}$ from $t_1 = f(z_1)/z_0$ *to* $t_2 = f(z_2)/z_0$. With the use of second part of equation (45), let choose γ' to be a segment of γ which joins t_1 and t_2 where $|t_2'| = 2|t_1'|$ and $1 \in A(|t_1'|, 2|t_2'|)$then $t_1', t_2' \in A(\frac{1}{2}, 2)$, therefore the density of hyperbolic metric on $\mathbb{C}\setminus\{0,1\}$ is bounded below on $A(\frac{1}{2}, 2)$ by absolute constant $2\delta > 0$ (say). Hence$\rho_{A_0}(z_1,z_2) \geq \rho_{\mathbb{C}\setminus\{0,1\}}(t_1', t_2') \geq \delta$ which is contradicts the fist inequality in (44). Therefore our assumption $z_0 \in \bar{A}(|f(z_1)|, |f(z_2|)\setminus f(A_0)$ is wrong. Therefore $f(A(R,R')) \supset \bar{A}(|f(z_1)|, |f(z_2)|)$, hence theorem is proved. □

Theorem 3.5. *Let f be a transcendental function. There exists $R_0 = R_0(f) > 0$ and an absolute constant $K > 1$ such that if*

$$0 \notin f^n(a(R,R')),\ for\ some\ n \in \mathbb{N} \tag{46}$$

where

$$R'/R \geq K^2\ and\ R \geq R_0 \tag{47}$$

then

$$f^n(A(R,R')) \supset A(M(R,f^n), M^n(R'/K,f)) \tag{48}$$

Proof. Let us consider $A(R,R')$ be an annulus which satisfies the (46) and (47) for some $K > 4$. We take the two points z_1, z_2 such as which satisfies

$$|z_1| = |z_2| = 4R'/K,\ |f^n(z_2)| = M(4R'/K, f^n)\ and\ |f^n(z_1)| = m(4R'/K, f^n).$$

Now we consider different cases that might arise. First, suppose that

$$|f^n(z_1)| \geq \frac{1}{2}|f^n(z_2)| = \frac{1}{2}M(4R'/K, f^n).$$

Since we know that $\partial f^n(A_0) \subset f^n(\partial A_0)$ and $R' > 4R'/K$, we have

$$|f^n(A_0)| \supset A\left(M(R,f^n), \frac{1}{2}M(4R'/K, f^n)\right). \tag{49}$$

Second, suppose that

$$M(4R'/K, f^n) = |f^n(z_2)| \geq 2|f^n(z_1)|.$$

By assumption, we have $4RK \leq 4R'/K < R'$ as if K is sufficiently large absolute constant, then $\rho_(A_0)(z_1, z_2) < \delta$, where δ is a constant given in previous theorem and, thus by the previous theorem

$$f^n(A_0) \supset \bar{A}(|f^n(z_1)|, |f^n(z_2)|) = \bar{A}(|f^n(z_1)|, M(4R'/K, f^n)). \tag{50}$$

If $|f^n(z_1)| \leq M(R, f^n)$, from (50), we have

$$f^n(A_0) \supset A(M(R, f^n), M(4R'/K, f^n)). \tag{51}$$

Now from equation (49) and (51) and using the properties of maximum modulus, that is $\log M(2r, f^n) \geq (1 + \log 2/logr) \log M(r, f^n)$ *and* $M((1+\epsilon), f^n) \geq M^n(r, f)$ where $\epsilon > 0$, we get

$$\frac{1}{2}M(4R'/K, f^n) \geq \frac{1}{2}M^n(2R'/K, f) \geq \frac{1}{2}M^n(R'/K, f)$$

for sufficiently large R'.

Hence by equation (51) we have $f^n(A(R, R')) \supset A(M(R, f^n), M^n(R'/K, f))$ and proof is complete. □

3.3 Introducing a Positive Harmonic Function

In this section we study a technique for wandering domains. Here we see that for each multiply connected wandering domain U there is positive harmonic function h which is defined in U.

Theorem 3.6. *Let f be transcendental entire function with a multiply connected wandering domain U and let $z_0 \in U$. Then*

$$h(z) = \lim_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0)|} \tag{52}$$

defines a non-constant positive harmonic function $h : U \to \mathbb{R}$, with $h(z_0) = 1$.

Let $z_0' \in U$ then the function denoted by h' and defined

$$h'(z) = \lim_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0')|} \implies h'(z) = \lim_{n\to\infty} \left\{ \frac{\log|f^n(z)|}{\log|f^n(z_0)|} \cdot \frac{\log|f^n(z_0)|}{\log|f^n(z_0')|} \right\}$$

$$\implies h'(z) = \lim_{n\to\infty}\left\{\frac{\log|f^n(z)|}{\log|f^n(z_0)|}\right\}\cdot\lim_{n\to\infty}\left\{\frac{\log|f^n(z_0)|}{\log|f^n(z_0')|}\right\} = h(z)\cdot\frac{1}{h(z_0')}$$

Thus it is clear that the function h, defined in (52), depends on z_0. If we replace z_0 by the another point $z_0 \in U$, then from above discussion, the resultant function is just h scaled by the positive factor $\frac{1}{h(z_0')}$.
Let U is domain, $U_n = f^n(U)$, $A(r,R) = \{z : r < |z| < R\}$ and $\bar{A}(r,R) = \{z : r < |z| < R\}$.

Theorem A. *Let f be a transcendental entire function with a multiply connected wandering domain U. If $A \subset U$ is a domain that contains a closed curve that is not null-homotophic in U, then, for sufficiently large $n \in \mathbb{N}$,*

$$U_n \supset f^n(A) \supset A(r_n, R_n)$$

where $R_n/r_n \to \infty$ *as* $n \to \infty$.

Let f be a transcendental entire function with multiply connected wandering domain U, let $z_0 \in U$ and, for $n \in \mathbb{N}$ let $r_n = |f(z_0)|$. In order prove the above theorem let consider the sequence of functions (h_n), where h_n defined by

$$h_n(z) = \frac{|f^n(z)|}{|f^n(z_0)|} \qquad z \in \bar{U},\ n \in \mathbb{N} \tag{53}$$

Without loss of generality we can assume that $f^n(z) > 1$ for $z \in \bar{U},\ n \in \mathbb{N}.$ and hence h_n is a positive harmonic with a continuous extension to $\bar{U}$. From the Harnack's theorem [Thm 18, Page-11 in [6]] and $h_n(z_0) = 1,\ for\ n \in \mathbb{N}$, there exist a sequence (n_k) such that h_{n_k} converges locally uniformly to a function h in U i.e;

$$h(z) = \lim_{k\to\infty} h_{n_k} \tag{54}$$

where h is positive harmonic function on U. In order to prove the above theorem, we have to show that h is non-constant and the whole sequence (h_n) converges to h in U. Therefore firstly we prove non-constant property.

Lemma 3.2. *Let h be the harmonic function on U defined by* (54). *Then h is non-constant.*

Proof. From Theorem A and lemma 3.1, m is sufficiently large, there exist $z_1, z_2 \in U$ such that

$$|f^n(f^m(z_2))| \geq M^n(|2f^m(z_1)|, f) \geq M(|2f^m(z_1)|, f^n) \qquad for\ n \in \mathbb{N}$$

and

$$|f^n(f^m(z_1))| \leq M(|2f^m(z_1)|, f^n) \qquad for\ n \in \mathbb{N}$$

Now

$$\liminf \frac{h_{n+m}(z_2)}{h_{n+m}(z_1)} = \liminf_{n\to\infty} \frac{\log|f^{n+m}(z_2)|}{\log|f^{n+m}(z_1)|}$$

using above inequalities we get

$$\liminf \frac{h_{n+m}(z_2)}{h_{n+m}(z_1)} \geq \frac{\log M(2|f^m(z_1)|, f^n)}{\log M(|f^m(z_1)|, f^n)}$$

$$\liminf \frac{h_{n+m}(z_2)}{h_{n+m}(z_1)} \geq \frac{\log M(2|f^m(z_1)|, f^n)}{\log M(|f^m(z_1)|, f^n)} > 1 \qquad using\ equation\ (29).$$

This implies $h(z_2) > h(z_1)$, since $z_1,\ z_2$ were arbitrary points in U. Therefore h is non-constant. □

Now second claim is to show the whole sequence h_n converges to h *in* U. For proving the second claim, firstly we prove two lemmas. The first lemma concerns the sequence

$$g_n(z) = \frac{\log f^n(z)}{\log r_n}, \qquad z \in V,\ n \in \mathbb{N} \tag{55}$$

where $V \subset U$ is a simply connected domain with z_0, the branch of logarithm is chosen so that $|\arg f^n(z_0)| \leq \pi$. Then we can see that g_n is analytic in V and $\Re g_n = h_n$.

Lemma 3.3. *Let $V \subset U$ be a simple connected domain with $z_0 \in V$, and let h_n, h and g_n be the function defined by (52), (53) and (55) respectively. If h_{m_k} is any subsequence of h_n such that $h_{m_k} \to h$ locally uniformly in U. Then*

(a) there exists an analytic function g on V with $\Re g = h$ such that $g_{m_k} \to g$ locally uniformly in V and

(b) for any continuum $C \subset V$, there exists $\epsilon > 0$ such that for large k

$$U_{m_k} \supset A\left(\min_{z\in C} |f^{m_k}(z)| r_{m_k}^{-\epsilon/8}, \max_{z\in C} |f^{m_k}(z)| r_{m_k}^{\epsilon/8}\right).$$

Proof. As f is a transcendental entire function, so (h_n) is locally uniformly bounded by the Harnack's inequality [6] and $h_n(z_0) = 1$ for $n \in \mathbb{N}$, since $\mathfrak{R} g_n = h_n$, therefore by Montel's Theorem, (g_n) is a normally family in V.

Let v be the harmonic conjugate of h on V, so that $v(z_0) = 0$, we get

$$g(z) = h(z) + iv(z)$$

Then g is analytic in V, we have

$$g_n(z) - g(z) = h_n(z) - h(z) + i(arg \log f^n(z)/ \log r_n - v(z)). \tag{56}$$

Now choose $\rho > 0$ such that $\{z : |z - z_0| \leq 2\rho\} \in V$. By the Borel-Caratheodory inequality, we have

$$\max_{|z-z_0|=\rho} |g_n(z) - g(z)| \leq 2 \max_{|z-z_0|=2\rho} |h_n(z) - h(z)| + 3|g_n(z_0) - g(z_0)|, \qquad for\ n \in \mathbb{N}. \tag{57}$$

As $h_{m_k} \to \infty$ locally uniformly in U, $\arg f^n(z_0) \leq \pi$ for $n \in \mathbb{N}$, and $v(z_0)$, then from equations (56) and (57), we obtained $g_{m_k} \to g$ uniformly in $\{z : |z - z_0| \leq \rho\}$. Since (g_n) is normal in V, then by Vitali's theorem $g_{m_k} \to g$ locally uniformly in V. Hence part (a) proved.

Now in order to prove of part (b), let us consider a continuum $C \in V$ and for $\delta > 0$, let C_δ be a nbd of C, there exists $\epsilon > 0$ such that

$$g(C_\delta) \supset g(C)_\epsilon.$$

Therefore large k we have, using part (a),

$$g_{m_k}(C_\delta) \supset g(C)_{\epsilon/2} \supset g_{m_k}(C)_{\epsilon/4}$$

and hence

$$\begin{aligned} U_{m_k} \supset f^{m_k}(C_\delta) &= \exp\left[(\log r_{m_k}) g_{m_k}(C_\delta)\right] \\ &\supset \exp\left[(\log r_{m_k}) g_{m_k}(C_\delta)\right] \\ &\supset A\left(\min_z \in C|f^{m_k}(z)|r_{m_k}^{-\epsilon/8}, \max_z \in C|f^{m_k}(z)|r_{m_k}^{\epsilon/8}\right) \end{aligned}$$

provided that $(\epsilon/8)\log r_{m_k} > \pi$. Hence part (b) proved. □

Lemma 3.4. *Let $z \in U$ and, for $n \in \mathbb{N}$, let h_n be the function defined by* (53) *and $\delta_n = 1/\sqrt{\log r_n}$. There exists $N \in \mathbb{N}$ such that*

(a) if

$$U_n \supset A(r_n^{1-2\pi\delta_n}, r_n^{1+2\pi\delta_n}) \qquad for\ some\ n \geq N$$

then

$$r_{m+n} \geq M(r_n, f^m)^{1-\delta_n} \qquad for\ m \in \mathbb{N}$$

(b) if, for some $n \geq N$

$$h_n(z) \geq 1 \qquad and\ \ U_n \supset A(r_n^{h_n(z)-2\pi\delta_n}, r_n^{h_n(z)+2\pi\delta_n}), \tag{58}$$

then

$$h_{n+m}(z) \geq h_n(z)(1-2\delta_n) \qquad for\ m \in \mathbb{N}$$

(c) if, for some $n \geq N$

$$h_n(z) \leq 1 \qquad and\ \ U_n \supset A(r_n^{h_n(z)-2\pi\delta_n}, r_n^{h_n(z)+2\pi\delta_n}), \tag{59}$$

then

$$h_{n+m}(z) \leq h_n(z)(1-2\delta_n) \qquad for\ m \in \mathbb{N}$$

(d) if $z' \in U$ and, for some $n \geq N$,

$$h_n(z)/h_n(z') \geq 1 \qquad and\ \ U_n \supset A(r_n^{h_n(z)-2\pi\delta_n}, r_n^{h_n(z)+2\pi\delta_n}), \tag{60}$$

then

$$h_{n+m}(z)/h_{n+m}(z') \geq (1-\delta_n)h_n(z)/h_n(z') \qquad for\ m \in \mathbb{N}$$

Proof. Let us assume that the inequality (58) is satisfies for some $z \in U$, $n \in \mathbb{N}$. Since $|f^m| > 1$ in U_n, for all $m \in \mathbb{N}$, and by the part (b) of Theorem 3.2 applied to $g = f^m$, for $m \in \mathbb{N}$ and sufficiently large n, we get

$$\log|f^m(f^n(z))| \geq (1-\delta_n)\log M(|f^n(z)|, f^m). \tag{61}$$

Putting $z = z_0$ in (61)

$$\log|f^m(f^n(z_0))| \geq (1-\delta_n)\log M(|f^n(z_0)|f^m).$$

Since $h_n(z_0) = 1$ and $r_n = |f^(z_0)|$,therefore

$$r_{m+n} \geq M(r_n, f^m)^{1-\delta_n} \qquad for\ m \in \mathbb{N}.$$

Hence part (a) is complete.

Proof of part (b)- let, from (52), $\log|f^n(z)| = h_n \log r_n$, for $n \in \mathbb{N},\ z \in U$. We assume that (58) satisfied for some $z \in U,\ n \in \mathbb{N}$, for n is sufficiently large and from (61), then we have, for $m \in \mathbb{N}$

$$h_{n+m}(z)\log r_{n+m} = \log|f^m(f^n(z))| \geq (1-\delta_n)\log M(r_n^{h_n(z)}, f^m).$$

From Theorem3.1 we obtain

$$h_{n+m}(z)\log r_{n+m} \geq (1-\delta_n)h_n(z)\log M(r_n, f^m) = (1-\delta_n)h_n(z)\log r_{n+m}$$

this implies

$$h_{n+m}(z)\log r_{n+m} \geq (1-\delta_n)h_n(z).$$

Hence part (b) is proved.

To prove part (c) we assume that (59) is satisfied for some $z \in,\ n \in \mathbb{N}$. Then for $m \in \mathbb{N}$ and sufficiently large n, we get

$$h_{n+m}(z)\log r_{n+m} = \log|f^m(f^n(z))| \leq \log M(|f_n(z)|, f^m) \qquad using; Theorem 3.1$$

$$= \log M(r_n^{h_n(z)}, f^m) \leq h_n(z)\log M(r_n, f^m).$$

By part (a), we obtain

$$h_{n+m}\log r_{n+m} \leq h_n(z)\log r_{n+m}/(1-\delta_n) \leq h_n(z)\log r_{n+m}.(1+2\delta_n)$$

this implies

$$h_{n+m}(z)\log r_{n+m} \leq h_n(z)(1+2\delta_n).$$

Thus part (c) completed.

To prove part (d), let us suppose that (60) is satisfied for some $z, z' \in U,\ n \in \mathbb{N}$, let us consider

$$\frac{h_{n+m}(z)}{h_{n+m}(z')} = \frac{\log f^{n+m}(z)}{\log f^{n+m}(z')}$$

from Theorem 3.2 part (b) and Theorem 3.1 with $r = |f^n(z')|$ *and* $r^c = |f^n(z)|$ we have

$$\begin{aligned}
&\geq (1-\delta_n)\frac{\log M(|f^n(z)|, f^m)}{\log M(|f^n(z')|, f^m)} \\
&\geq (1-\delta_n)\frac{\log |f^n(z)|}{\log |f^n(z')|} \\
&\geq (1-\delta_n)\frac{h^n(z)}{h^n(z')}
\end{aligned}$$

Hence part (d) proved. □

Proof of Theorem 3.6. For the function h_n defined in (52), we know that there exists a subsequence n_k such that the functions h_{n_k} converges uniformly to non-constant positive harmonic function h. Thus by the hypothesis of Lemma 3.3, its holds for the sequence (n_k). Then by the Lemma 3.3 part(b) it is clear that we can apply Lemma 3.4 part (b) and part (c) to U_{n_k}, provided k is sufficiently large, this gives that the whole sequence h_n converges to h locally uniformly in U. Hence theorem 3.6 is proved. □

Consequently we shall see that the another result, that is the image of a multiply connected wandering domain must contain annuli which is much larger than, that given in Theorem A. In fact the image of any open set in such a domain must eventually contains so large annuli.

Theorem 3.7. *Let f be a transcendental entire function with a multiply connected wander domain U. For each $z_0 \in U$ and each open set $D \subset U$ containing z_0 there exists $\alpha > 0$ such that, for sufficiently large $n \in \mathbb{N}$,*

$$U_n \supset f^n(D) \supset A(|f^n(z_0))|^{1-\alpha}, |f^n(z_0)|^{1+\alpha}$$

Proof. Proof of this theorem will be obtain directly from Lemma 3.3 with using Theorem 3.6, as we have seen in part (b) of Lemma3.3 that for simply connected domain $V \in U$ containing z_0, and if any subsequence h_{m_k} of h_n such that its converges locally uniformly to h *in* U, for any continuum $C \subset V$, there exists $\epsilon > 0$ such that for large k,

$$U_{m_k} \supset A\left(\min_{z\in C} |f^{m_k}(z)| r_{m_k}^{-\epsilon/8}, \max_{z\in C} |f^{m_k}(z)| r_{m_k}^{\epsilon/8}\right).$$

Now apply above statement for $C = \{0\}$, we get

$$U_n \supset f^n(D) \supset A(|f^n(z_0))|^{1-\alpha}, |f^n(z_0)|^{1+\alpha}$$

its proves theorem. $\square$

3.4 Dynamics in a Multiply Connected Wandering Domain

In this section we study the dynamics in a multiply connected wandering domain. We see that a multiply connected wandering domains of entire functions have a strong property that $\log R_n / \log r_n \to \infty$ *as* $n \to \infty$ but this is not true in general because its $\sup_{n\to\infty} \log R_n / \log r_n$ may be arbitrary close to 1.
Theorem 3.6 is able us to provide a detailed description of the dynamical behavior in any multiply connected wandering domain, which is parallel to the classical results on the behavior in attracting and parabolic Fatou components. For any multiply connected wandering domain of f we can identify a simple 'absorbing set ' for f, that is a set in which the iterates of all point in U eventually lie in this, in that the dynamical behavior of f is well-behaved.

Let f be a transcendental entire function with a multiply connected wandering domain U, let $z_0 \in U$ and there exists $\alpha > 0$ such that, for large $n \in \mathbb{N}$, the maximum annulus in U_n, with centred at 0, which contains $f(z_0)$ is of the form

$$B_n = A(r_n^{a_n}, r_n^{b_n}) \qquad where\ \ r_n = |f^n(z_0)|,\ \ 0 < a_n < 1 - \alpha < 1 + \alpha < b_n \tag{62}$$

These annuli B_n have many useful properties. Here we show that the union of the annuli works as an absorbing set indeed, we can specify an absorbing set for f consisting of somewhat smaller annuli.

Theorem 3.8. *Let f be the transcendental entire function with multiply connected wandering domain U and let $z_0 \in U$. Then, for each compact subset C of U, there exists $N \in \mathbb{N}$ such that*

$$f^n(C) \subset B_n \qquad for\ n \geq N.$$

In fact

$$f^n(C) \subset C_n \qquad for\ n \geq N,$$

where

$$C_n = A\left(r_n^{a_n+2\pi\delta_n}, r_n^{b_n(1-3\pi\delta_n)}\right) \qquad with\ \delta_n = 1/\sqrt{\log r}. \tag{63}$$

Proof. Let us consider the harmonic function h_n, which defined as previous,and defined the limit function $h(z)$ as

$$h(z) = \lim_{n\to\infty} h_n(z) \qquad for\ z \in U \tag{64}$$

which is a non-constant positive harmonic function in U.

Now this theorem states that, if C is a compact subset of U, then $f^(C) \subset C_n$, for large value of $n \in \mathbb{N}$. Therefore in oder to prove of theorem we cover C by a finite number of closed discs $D_1, D_2, \ldots, D_p$ in U, and then, for $j = 1, 2, \ldots, p$ join D_j to z_0 by a simple curve Lj in U. From (64) it clear that we can apply Lemma 3.3 part (b) for each continuum $L_j \cup D_j$ $j = 1, 2, \ldots, p$, with appropriate simply connected domain $V_j \subset U$ containing $D_j \cup L_j$ to show that there exists $\epsilon > 0$ such that

$$U_n = f^n(U) \supset A\left(\min_{z\in C'} |f^(z)| r_n^{-\epsilon/8}, max_{z\in C'} |f^n(z)| r_n^{\epsilon/8}\right) \qquad for\ large\ n \in \mathbb{N}$$

where $C' = \cup_{j=1}^{p}(D_j \cup L_j)$. Hence as $z_0 \in C'$ *and* B_n is the largest annulus in U containing $f^n(z_0)$, we have

$$f^n(C) \subset A(r_n^{a_n+\epsilon/8}, r_n^{b_n-\epsilon/8}). \tag{65}$$

Now let us consider $H = \max_{z\in C} h(z)$. Here two cases occurs first if $b_n \leq 2H$, then for sufficiently large n we have

$$a_n + \epsilon/8 > a_n + 2\pi\delta_n\ and\ b_n - \epsilon/8 < b_n(1 - 3\pi\delta_n).$$

By the (65) we get $f^n(C) \subset C_n$.

Second if $b_n > 2H$ then for sufficiently large n we have $a_n + \epsilon/8 > a_n + 2\pi\delta_n$ and

$$b_n(1 - 3\pi\delta_n) > b_n/2 > H = \max_{z\in C} h_n(z) = \max_{z\in C} \frac{\log|f^n(z)|}{\log r_n},$$

so again by (65) we get $f^n(C) \subset C_n$. Thus this completes the proof. $\square$

Our next result is informed that, in C_n the minimum modulus is very close to the maximum modulus. This is a key result of the Theorem 3.2 and the following result follows immediately by taking $g = f^m,\ a = a_n,\ b = b_n\ and\ \delta = \delta_n$ in Theorem 3.2.

Theorem 3.9. *Let f be transcendental entire function with a multiply connected wandering domain U,let $z_0 \in U$ and, for large nßℕ, let r_n, a_n and b_n be defined as in* (62)

$$m(\rho, g) > 1 \qquad for\ \rho \in (r^a, r^b). \tag{66}$$

(a) if $m \in \mathbb{N}$ and $\epsilon \in (\pi/\log r_n, (b_n - a_n)/2)$, where $n \in \mathbb{N}$ is sufficiently large, then

$$\log m(\rho, f^m) \geq \left(1 - \frac{2\pi}{\epsilon \log r_n}\right) \log M(\rho, f^m) \qquad for\ \rho \in [r_n^{a_n+\epsilon}, r_n^{b_n-\epsilon}].$$

(b) In particularly, if $\delta_n = 1/\sqrt{\log r_n}$, then, for $m \in \mathbb{N}$ and large $n \in \mathbb{N}$

$$\log m(\rho, f^m) \geq (1 - \delta_n) \log M(\rho, f^m) \qquad for\ \rho \in [r_n^{a_n+2\pi\delta_n}, r_n^{b_n-2\pi\delta_n}].$$

The proof of above theorem is follows immediately from theorem 3.2 as describe before the statement of the theorem. Now in order to the description of the properties of function h, we represent the next result.

Theorem 3.10. *Let f be a transcendental entire function with a multiply connected wandering domain U, let $z_0 \in U$ and for large $n \in \mathbb{N}$, let $r_n, a_n, b_n, \delta_n, B_n$ and C_n be defined as in* (62) *and* (63)*. There exists $N \in \mathbb{N}$ such that for $n \geq N$ and $m \in \mathbb{N}$, the preimage under f^m of a $A\left(M(r^{a_n}, f^m), M(r^{(b_n-2\pi\delta_n)(1-\delta_n)})\right)$ has a doubly connected component $A_{n,m}$ such that*

$$C_n \subset A_{n,m} \subset B_n$$

and f^m has no critical points in $A_{n,m}$. Moreover, for $n \geq N$ and $m \in \mathbb{N}$, we can write $f^m(z) = P_{n,m}(\phi_{n,m}(z))\ z \in A_{n,m}$ where
(a) $\phi_{m,n}$ is a conformal map on $A_{n,m}$ which satisfies

$$|z|^{1-\delta_n-2\pi\delta_n \log r_n/log|z|} \leq |\phi_{n,m}(z)| \leq |z|^{1+2\delta_n}, \qquad for\ r_n^{a_n+4\pi\delta_n} \leq |z| \leq r_n^{b_n(1-5\pi\delta_n)}; \tag{67}$$

(b) $P_{n,m}(z) = q_{n,m} z^{d_{n,m}}$*, where* $q_{n,m} > 0$ *and the degree* $d_{n,m}$ *is the number of zeros of* f^m *in* $\{z : |z| < r_n\}$*, which satisfies*

$$(1 - 2\delta_n)\frac{\log M(r_n, f^m)}{\log r_n} \leq d_{n,m} \leq \frac{\log M(r_n, f^m)}{(1 - a_n)\log r_n}.$$

Theorem 3.11. *Let* f *be a transcendental entire function with a multiply connected wandering domain* U*, let* $z_0 \in U$ *and, for large* $n \in \mathbb{N}$*, let* $r_n, a_n, b_n, \delta_n, B_n$ *and* C_n *be defined as in* (62) *and* (63)*. There exists an absolute constant* $K > 1$ *such that, for* $m \in \mathbb{N}$ *and large* $n \in \mathbb{N}$*,*

(a)

$$M(r_n, f^m) \geq r_{n+m} \geq M(r_n, f^m)^{1-\delta_n}$$

(b)

$$f^m(B_n) \cap B_{n+m} \supset A\left(M(r_n^{a_n}, f^m), M(r_n^{b_n}/K, f^m)\right) \supset A(r_{n+m}^{a_n+\delta_n}, r_{n+m}^{b_n-\delta_n})$$

(c)

$$f^m(C_n) \subset B_{n+m}.$$

Proof. (a) We know that $r_n = |f^n(z_0)|$, for $n \in \mathbb{N}$ and so $r_{n+m} = |f^{n+m}(z_0)|$ and also we know that $f(z) \leq M(r, f)$, therefore it is clear that $r_{n+m} = M(r_n, f^m)$. Now for large $n \in \mathbb{N}$ and $m \in \mathbb{N}$, from Theorem 3.10 part (b) that is

$$\log m(\rho, f^m) \geq (1 - \delta_n)\log M(\rho, f^m) \qquad \textit{for } \rho \in [r_n^{a_n+2\pi\delta_n}, r_n^{b_n-2\pi\delta_n}]$$

$$\implies m(\rho, f^m) \geq M(\rho, f^m)^{1-\delta_n} \qquad \textit{for } \rho \in [r_n^{a_n+2\pi\delta_n}, r_n^{b_n-2\pi\delta_n}]$$

since $m(\rho, f) \leq |f(z)|\ \forall z \in U$, thus this implies that $r_{n+m} \geq M(\rho, f^m)^{1-\delta_n}$. Combining these two inequalities, prove part (a) is completed.

(b) Now we recall from Theorem 3.7 that there exists $\alpha > 0$ such that

$$a_n \leq 1 - \alpha \ \textit{ and } b_n \geq 1 + \alpha \ \textit{ for large } n \in \mathbb{N}. \tag{68}$$

Hence $b_n - a_n \geq (1 + \alpha) - (1 - \alpha) = 2\alpha$ this implies $r_n^{b_n - a_n} \geq r_n^{2\alpha}$. Thus by Theorem 3.5 there exists an absolute constant $K > 0$ such that, for large $n \in \mathbb{N}$ and for $m \in \mathbb{N}$,

$$f^m(B_n) \supset A\left(M(r_n^{a_n}, f^m), M^m(r_n^{b_n}/K, f)\right) \tag{69}$$

$$\supset A\left(M(r_n^{a_n}, f^m), M(r_n^{b_n}/K, f^m)\right)$$
$$\supset A\left(M(r_n^{a_n}, f^m), M(r_n^{b_n}/K, f^m)\right).$$

From Theorem 3.1,part (a) of this result and (68), we have, for $m \in \mathbb{N}$ and large $n \in \mathbb{N}$,

$$M(r_n^{a_n}, f^m) \leq M(r_n, f^m)^{a_n} \leq r_{n+m}^{a_n/(1-\delta_n)} \leq r_{n+m}^{a_n(1+\delta_n/(1-\alpha))} \leq r_{n+m}^{a_n+\delta_n}. \tag{70}$$

Again from (68), if n is sufficiently large, then $b_n - \delta_n > 1$. So by Theorem 3.1 and part (a) of this result, for $m \in \mathbb{N}$ and large $n \in \mathbb{N}$,

$$M(r_n^{b_n}/K, f^m) \geq M(r_n^{b_n-\delta_n}, f^m) \geq M(r_n, f^m)^{b_n-\delta_n} \geq r_{n+m}^{b_n-\delta_n}. \tag{71}$$

From (69), (70) and (71), we get that, for $m \in \mathbb{N}$ and large $n \in \mathbb{N}$,

$$f^m(B_n) \supset A\left(M(r_n^{a_n}, f^m), M^m(r_n^{b_n}/K, f)\right) \supset A(r_{n+m}^{a_n+\delta_n}, r_{n+m}^{b_n-\delta_n})$$

with $a_n + \delta_n < b_n - \delta_n$ and, we have

$$B_{n+m} \supset A\left(M(r_n^{a_n}, f^m), M^m(r_n^{b_n}/K, f)\right). \tag{72}$$

Hence combining these above equation gives the result Part (b).

(c) By Theorem 3.11 and the definition of the annulus $A_{n,m}$ in Theorem3.11, obtained, for large $n \in \mathbb{N}$ and $m \in \mathbb{N}$,

$$f^m(C_n) \subset f^m(A_{n,m}) = A\left(M(r_n^{a_n}, f^m), M^m(r_n^{(b_n-2\pi\delta_n)(1-\delta_n)}, f^m)\right).$$

Thus, by (72), for large $n \in \mathbb{N}$ and $m \in \mathbb{N}$, we have $f^m(C_n) \subset B_{n+m}$. This completes the proof. $\square$

In this section we have seen that how the sequence of annuli (B_n) and (C_n) relate to each other i.e., we have seen

$$f^m(C_n) \supset A(r_{n+m}^{a_n+\delta_n}, r_{n+m}^{b_n-\delta_n}) \qquad and \qquad f^m(C_n) \subset B_{n+m} \tag{73}$$

These above results are applicable to describe further geometric properties of the Fatou components U_n. From first part in (73), we can see that the sequence of exponent (a_n) and (b_n) in the definition of annuli B_n are both convergent, with possibility that $b_n \to \infty\ as\ n \to \infty$. We can prove that, for large n, the annulus B_n forms a relatively largest subset of U_n. To make this precise we define-

Definition : the outer boundary component of a bounded domain U to be the boundary of the unbounded component of $\mathbb{C}\backslash U$, which is denoted by $\partial_{out}U$ and the inner boundary component of U to be the boundary of the component of $\mathbb{C}\backslash U$ that contains 0 if there is one, denoted by $\partial_{inn}U$.

3.5 Geometric Properties of Multiply Connected Wandering Domains

In this section we will consider the geometric properties of multiply connected wandering domains of an entire function and prove the following theorem. First we recall that for any bounded domain U the outer boundary component of U is denoted by $\partial_{out}U$ and the inner boundary component of U is denoted by $\partial_{inn}U$. Now let U be a multiply connected wandering domain. Since $f : U \to U_n$ is a proper map, each boundary component of U is mapped by f^n, $n \in \mathbb{N}$, to a boundary component of $U_n = f^n(U)$. In fact, since $\partial U_n \subset f^n(\partial U)$, the outer boundary component of U maps to the outer boundary component of U_n. As there exists $N \in \mathbb{N}$ such that the image of domains $U_n = f^n(U)$, $n \geq N$, surrounding 0, U_n has an inner boundary component for $n \geq N$, and f maps the inner boundary component of such a U_n to the inner boundary component of U_{n+1}. Therefore our next results involves in the following theorem.

Theorem 3.12. *Let f be transcendental entire function with a multiply connected wandering domain U, for large $n \in \mathbb{N}$, and let r_n, a_n and b_n be define as in previous results.*

(a) Then, as $n \to \infty$,

$$a_n \to a \in [0,1) \text{ and } b_n \in (0,\infty]$$

(b) For large $n \in \mathbb{N}$, let $\underline{a}_n$ and$\underline{b}_n$ denot the smallest values such that

$$\{z : |z| = r_n^{\underline{a}_n}\} \text{ and } \{z : |z| = r_n^{\underline{b}_n}\}$$

meets $\partial_{inn}U$ and $\partial_{out}U$ respectively, and let $\bar{a}_n$ and $\bar{b}_n$ denote the largest such values. Then

(i) $b \geq \underline{b}_n \geq b_n$ *and* $b/a \geq \underline{b}_n/\underline{a}_n \geq b_n/a_n$, *for sufficiently large* n;

(ii) $\underline{b}_n \to b, \underline{a}_n \to a$ *and* $\bar{a}_n \to a$ *as* $n \to \infty$

Proof. Let $B_n = A(r_n^{a_n}, r_n^{b_n})$ be define already previously. Now here our aim to show that the sequences (a_n) and (b_n) are convergent and these converges to, we want to call its a and b, respectively.

From Theorem 3.11 part (a) , for $m \in \mathbb{N}$ and large $n \in \mathbb{N}$ we obtain

$$a_{n+m} \leq a_n + \delta_n \text{ and } b_[n+m] \geq b_n + \delta_n.$$

Using (68), wet get

$$a_n \to a \in [0,1[\text{ and } b_n \to b \in]1,\infty] \text{ as } n \to \infty.$$

Thus part (a) is completed. To prove part (b), by the definition of $\underline{b}_n, \underline{a}_n, \bar{a}_n$ *and* $\bar{b}_n$, it is clear that $\underline{b}_n \geq b_n$ and $\underline{b}_n/\underline{a}_n \geq b_n/a_n$ for $n \in \mathbb{N}$.

Claim to show $b \geq \underline{b}_n$ *and* $b/a \geq \underline{b}_n/\underline{a}_n$ for sufficiently large $n \in \mathbb{N}$.

For this we take $N \in \mathbb{N}$ too large that $r_n^{\underline{a}_n} \geq R$ for $n \geq N$ where R is the constant in Theorem 3.1. Then we also take $\epsilon > 0$ so small that $a_n + \epsilon < 1 < b_n - \epsilon$ for $n \geq N$, which occurs due to Theorem 3.1. Now fix $n \geq N$ and $m \in \mathbb{N}$, let w_m be a point which satisfying

$$|w_m| = r_n^{\underline{b}_n - \epsilon} \text{ and } |f^m(w_m)| = M(r_n^{\underline{b}_n - \epsilon}, f^m) \tag{74}$$

Then there exist w_0 and a sequence (m_k) such that

$$|w_0| = r_n^{\underline{b}_n - \epsilon} \text{ and } w_0 = \lim_{k \to \infty} w_{m_k}. \tag{75}$$

We claim that $w_0 \in U_n$. Assume it is not in U_n, then w_0 belongs to a bounded component of the complement of U_n, therefore there exists $M \in \mathbb{N}$ such that $|f^M(w_0)| < r_{n+M}$, since a component lies in the interior of a Jordan curve in U so by the theorem 3.8, it must map under f^m for large m to a component of $\mathbb{C} \backslash U_{n+m}$ that is surrounded by B_{n+m}. Thus for sufficiently large k, $|f^M(w_{m_k})| < r_{n+m}$ and hence

$$|f^{m_k}(w_{m_k})| < M(r_{n+M}, f^{m_k - M}) < M(r_n, f^{m_k}) < M(r_n^{\underline{b}_n - \epsilon}, f^{m_k}).$$

This contradicts to (74) and so w_0 must be in U_n as claimed. Now from (74), (75) and Theorem 3.8 we get, for k sufficiently large,

$$B_{n+m_k} \supset \left\{z : |z| = M(r_n^{\underline{b}_n-\epsilon}, f^{m_k})\right\}. \tag{76}$$

By Theorem 3.1 and Theorem 3.11 part (a), we obtain, for $k \in \mathbb{N}$,

$$M(r_n^{\underline{b}_n-\epsilon}, f^{m_k}) \geq M(r_n, f^{m_k})^{\underline{b}_n-\epsilon} \geq r_{n+m_k}^{\underline{b}_n-\epsilon}.$$

Using (76), k is sufficiently large, we have $b_{n+m_k} \geq \underline{b}_n - \epsilon$ and so $b \geq \underline{b}_n - \epsilon$, since ϵ can be chosen arbitrarily very small, this implies $b \geq \underline{b}_n$ as required. Next be the same value of n and $\epsilon > 0$ so small that $a_n + \epsilon < 1 < b_n - \epsilon$, let $z' \in U_n$ satisfy $|z'| = r_n^{\underline{a}_n+\epsilon}$. Then by Theorem 3.8

$$B_{n+m_k} \supset \{z : |z| = |f^m(z')|\} \quad for\ large\ m \in \mathbb{N}. \tag{77}$$

It follows from Theorem 3.1 that, for $k \in \mathbb{N}$,

$$M(r_n^{\underline{b}_n-\epsilon}, f^{m_k}) \geq M(r_n, f^{m_k})^{(\underline{b}_n-\epsilon)/(\underline{a}_n+\epsilon)} \geq |f^{m_k}(z')|^{(\underline{b}_n-\epsilon)/(\underline{a}_n+\epsilon)}. \tag{78}$$

From (76), (77) and Theorem3.8, the (3.5) implies that, if k is sufficiently large, then $b_{n+m_k}/a_{n+m_k} \geq (\underline{b}_n - \epsilon)/(\underline{a}_n + \epsilon)$ and so $b/a \geq \underline{b}_n/\underline{a}_n$ as $\epsilon \to 0$. This completes the proof of part (b) (i).

To prove part (b) (ii) from above early result give us, for large n, $b \geq \underline{b}_n \geq b_n$. Since $b_n \to b\ as\ n \to \infty$ this gives that $\underline{b}_n \to b\ as\ n \to \infty$. Similarly large n we have $b/a \geq \underline{b}_n/\underline{a}_n \geq b_n/a_n$ since $b_n/a_n \to b/a\ as\ n \to \infty$

$$\implies \underline{b}_n/\underline{a}_n \to b/a\ as\ n \to \infty.$$

Thus $\underline{a}_n \to a\ as\ n \to \infty$ because $\underline{b}_n \to b\ as\ n \to \infty$. we have $\underline{a}_n \leq \bar{a}_n \leq a_n$ for large n this implies that $\bar{a}_n \to a\ as\ n \to \infty$. This completes the proof. □

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